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KEY

TO

ELEMENTARY TEXT-BOOK

OF

MECHANICS

CONTAINING EXAMINATION QUESTIONS ON THE TEXT,
INTRODUCTORY EXAMPLES, AND SOLUTIONS OF
THE EXERCISES.

BY

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PREFACE.

THIS Key is designed to aid the student in an intelligent study of my *Text-Book of Mechanics.* It contains :—

(*a*) Examination Questions upon each chapter;
(*b*) Easy Examples illustrating the principles of the text;
(*c*) Full Solutions of all the Exercises.

Care has been taken to refer in all cases to the theorems and demonstrations made use of in working the exercises; and it is hoped that the student will not rest satisfied with merely solving the problems, but will acquire a thorough knowledge of the principles he employs in his work.

Occasionally different solutions of a problem are given, in order to illustrate the statements of the text, and to serve as examples of the methods that may be applied to similar questions.

The student should always attempt the solution of an exercise himself before consulting the Key, and he should try to solve it in two or more different ways.

J. J. D.

DUBLIN,
April, 1884.

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Elementary Text-Book of Mechanics.

CHAPTERS I. AND II.

EXAMINATION QUESTIONS.

1. What are the primary conceptions of the material universe?

2. State the respective sciences which treat of these fundamental ideas.

3. Give a definition of each of these sciences.

4. Define Kinematics, and point out its relation to Algebra and Geometry.

5. What is motion? What is velocity?

6. How is velocity divided? Define and illustrate uniform velocity and variable velocity.

7. How is variable velocity divided? Define acceleration.

8. State and prove the equation which expresses the relations between space, time, and velocity, when the velocity is *uniform.*

9. Give the three equations which express these relations when the velocity is *uniformly accelerated:*—(*a*), with an initial velocity; (*b*), when there is no initial velocity.

10. Show how these equations are obtained.

11. Prove the second of these formulae without making any assumption not self-evident.

12. What is the most important example in nature of a uniformly accelerated velocity?

13. Give the equations which express the relations between space, time, and velocity in the case of falling bodies.

14. When a body falls freely from rest in a vacuum, state and prove the propositions which express the relations between the spaces described and the times of falling.

[*] UNITS OF VELOCITY AND ACCELERATION.

15. Define unit of velocity, and unit of acceleration.

16. Show how the unit of velocity is changed when a change is made in the units of space and time.

17. Show how the unit of acceleration is changed by a change in the units of space and time.

18. How is the numerical measure of a quantity affected by a change in the unit?

19. How are the numerical measures of a velocity and an acceleration respectively affected by a change in the units of time and space?

20. Show that velocities and accelerations may be represented by lines.

INTRODUCTORY EXAMPLES.

(*a*) Three bodies have uniform velocities of 15 miles per hour, 120 yards per minute, and 10 yards per second, respectively: express these velocities in feet per second.

(1) 15 miles $= 15 \times 5280$ feet $\therefore$ the velocity of the first body $= 15 \times 5280$ feet per hour. But as there are in an hour 60×60 seconds, the velocity per second will be the $\frac{1}{60 \times 60}$th of that per hour $\therefore$ 15 miles per hour $= \frac{15 \times 5280}{60 \times 60}$ feet per second $= 22$ feet per second.

(2) 120 yards per minute $= \frac{120 \times 3}{60}$ feet per second $= 6$ feet per second.

(3) 10 yards per second $= 10 \times 3$ feet per second $= 30$ feet per $1''$.

(*b*) A body is observed to move in successive seconds with the velocities of 50 feet, 40 feet, 30 feet ..... per second: what is the acceleration?

The body loses a velocity of 10 feet per second every second, and therefore (Art. 5) the acceleration $= -10$ feet per $1''$ per $1''$.

(*c*) A body moves with a uniform velocity of 12 feet per second: (1), what length of path will it describe in 5 seconds? (2), how long will it take to move over a distance of a mile?

(1) By Art. 6, $s = ut$ $\therefore$ length of path $= 12 \times 5 = 60$ feet.

(2) By Art. 6, $s = ut$ $\therefore$ $5280 = 12t$ $\therefore$ $t = \dfrac{5280}{12} = 440$ seconds.

(*d*) Two bodies moving with different uniform velocities pass over spaces of 120 feet and 150 yards respectively, the first in 4 seconds and the second in 2 minutes; find the velocity of each body.

By Art. 6, $s = ut$ $\therefore$ $u = \dfrac{s}{t}$ $\therefore$

(1) Velocity $= \dfrac{120}{4} = 30$ feet per second.

(2) Velocity $= \dfrac{150 \times 3}{2 \times 60} = 3\frac{3}{4}$ feet per second.

(*e*) One body moves with a uniform velocity of 100 yards per minute, and another with a uniform velocity of 200 centimetres per second: what space will each describe in 20 seconds?

By Art. 6, $s = ut$. Also 100 yards per minute $= \dfrac{100 \times 3}{60}$ feet per second.

$\therefore$ (1) Space $= \dfrac{100 \times 3}{60} \times 20$ feet $= 100$ feet.

(2) Space $= 200 \times 20$ centimetres $= 4000$ centimetres.

(*f*) A body has an initial uniform velocity of 120 feet per second, and it receives an acceleration of 4 feet per second in the same direction as the initial velocity: (1), what will be its velocity at the end of 3 seconds? (2), what space will it have described in that time?

(1) We employ equation (1) of Art. 11 with the + sign, since it is the equation connecting velocity and time with an initial velocity in the same direction as the acceleration.

$v = u + ft = 120 + 4 \times 3 = 132$ feet per second.

(2) We employ equation (2) of Art. 11, since it connects space and time.

$s = ut + \frac{1}{2}ft^2 = 120 \times 3 + \frac{1}{2} \times 4 \times 3^2 = 378$ feet.

(*g*) If in the foregoing question the acceleration were in the opposite direction to the initial velocity: (1), what would the velocity be in 10 seconds? (2), what would be its distance from the starting point in that time?

We employ the equations of Art. 11 with the – sign.

(1) $v = u - ft = 120 - 4 \times 10 = 80$ feet per second.

(2) $s = ut - \frac{1}{2}ft^2 = 120 \times 10 - \frac{1}{2} \times 4 \times 10^2 = 1000$ feet.

(*h*) If in Exam. (*f*), the body, possessing an initial velocity of 120 feet per second and receiving an acceleration in the same direction of 4 feet per second, move through a space of 500 feet, what will be its velocity?

We employ equation (6) of Art. 11, since it connects space and velocity.

$v^2 = u^2 + 2fs = 120^2 + 2 \times 4 \times 500 \therefore v = \sqrt{18400} = 135{\cdot}6$ feet per 1″.

(*i*) A body falls freely from rest: (1), how far will it fall in 5 seconds? (2), how far in 10 seconds?

We may employ equation (2) of Art. 14.

(1) $s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times 5^2 = 400$ feet.

(2) $s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times 10^2 = 1600$ feet.

Or, using the method of Art. 15 (3):—

(1) 5 seconds = 20 quarter seconds $\therefore$ space $= 20^2 = 400$ feet.

(2) 10 seconds = 40 quarter seconds $\therefore$ space $= 40^2 = 1600$ feet.

(*j*) [*]. How are the units of velocity and acceleration respectively affected when the unit of length is changed from 1 foot to 2 feet, and the unit of time from 1 second to 10 seconds?

Art. 17. (1) By the change in the unit of length alone, the unit of velocity is doubled, and by the change in the unit of time alone, it becomes one-tenth of its original amount, therefore with both changes the new unit of velocity $= \frac{2}{10}$ths $= \frac{1}{5}$th of former unit.

(2) By the change in the unit of length alone, the unit of acceleration is doubled, and by the change in the unit of time alone it becomes $\frac{1}{10 \times 10} = \frac{1}{100}$th its original value, therefore with both changes the new unit of acceleration $= \frac{2}{100}$ths $= \frac{1}{50}$th of the former unit.

(*k*) [*] A velocity is expressed by 20, and an acceleration by 4, when the units of space and time are respectively 1 foot and 1 second. If these units be changed as stated in the last question what will be the measure of the above velocity, and what will be the measure of the acceleration?

Art. 18. Since the new unit of velocity is $\frac{1}{5}$th the former, the measure of the velocity $20 = 20 \times 5 = 100$. And the measure of the acceleration $= 4 \times 50 = 200$.

Or—

A velocity of 20 feet per second $= \frac{20}{2}$ new units of length per second $= \frac{20}{2} \times 10$ units of length per 10 seconds $= 100$.

And 4 feet per second per second

$= \frac{4}{2} \times 10 \times 10$ units of length per 10″ per 10″ $= 200$.

Hence a velocity of 20 feet per second = a velocity of 100 units of 2 feet each per 10 seconds.

And an acceleration of 4 feet per 1″ per 1″ is the same as an acceleration of 200 units of 2 feet each per 10″ per 10″.

SOLUTIONS OF EXERCISES TO CHAP. II. Page 13.

1. There being no initial velocity we employ the equations of Art. 12.

(a) $v = ft = 20 \times 10 = 200$ feet per second.

(b) $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 20 \times 5^2 = 250$ feet.

(c) $v^2 = 2fs = 2 \times 20 \times 40 \therefore v = \sqrt{1600} = 40$ feet per 1″.

2. We employ the equation of Art. 12 connecting velocity and time.

$$v = ft \therefore t = \frac{v}{f} = \frac{120}{30} = 4 \text{ seconds.}$$

3. We employ the equation of Art. 12 connecting space and time.

$$s = \tfrac{1}{2}ft^2 \therefore t = \sqrt{\frac{2s}{f}} = \sqrt{\frac{2 \times 900}{8}} = 15 \text{ seconds.}$$

4. Since there is an initial velocity, and in the same direction as the acceleration, we use the equations of Art. 11 with the + sign.

(a) $v = u + ft = 100 + 10 \times 30 = 400$ feet per second.

(b) $s = ut + \frac{1}{2}ft^2 = 100 \times 5 + \frac{1}{2} \times 10 \times 5^2 = 625$ feet.

5. Using the same equations, but with the − sign.

(a) $v = u - ft = 100 - 10 \times 30 = -200$ feet per second.

(b) $s = ut - \frac{1}{2}ft^2 = 100 \times 5 - \frac{1}{2} \times 10 \times 5^2 = 375$ feet.

6. The question is equivalent to: "How far will the body move before its velocity becomes 0?" We therefore employ the equation of Art. 11 connecting velocity and space, and using the − sign.

$v^2 = u^2 - 2fs$. Now when the body comes to rest $v = 0 \therefore$

$0 = 20^2 - 2 \times 5 \times s \therefore 10s = 400 \therefore s = 40$ feet.

7. This question is equivalent to: "For what time will the body move before its velocity becomes 0?" We therefore employ the equation of Art. 11 connecting velocity and time, and using the − sign.

$v = u - ft \therefore 0 = 20 - 5t \therefore 5t = 20 \therefore t = 4$ secs.

8. The space described in the 6th second is equal to the space described in 6 seconds less by the space in 5 seconds. Using equation (2) Art. 12, $s = \frac{1}{2}ft^2 \therefore$ space in $6'' = \frac{1}{2}f \times 6^2$, and space in $5'' = \frac{1}{2}f \times 5^2$. $\therefore$ space in 6th second $= \frac{1}{2}f \times 6^2 - \frac{1}{2}f \times 5^2 = \frac{1}{2}f(6^2 - 5^2)$. But by the question, the space in 6th second $= 132$ feet $\therefore$

$$132 = \frac{1}{2}f(6^2 - 5^2) \therefore f = \frac{132 \times 2}{36 - 25} = 24 \text{ feet per } 1'' \text{ per } 1''$$

The whole space described in the 6 seconds is found from the same equation of Art. 12,

$$s = \frac{1}{2}ft^2 \therefore s = \frac{1}{2} \times 24 \times 6^2 = 432 \text{ feet.}$$

9. We employ equation (2) Art. 12, since it connects space and time, and taking the unit of acceleration = 1 yard per minute per minute, therefore the given acceleration = 80 of these units, and 30 seconds = $\frac{1}{2}$ a unit.

$$s = \frac{1}{2}ft^2 = \frac{1}{2} \times 80 \times (\tfrac{1}{2})^2 = 10 \text{ yards} = 30 \text{ feet.}$$

Or :—

Taking the unit of acceleration as 1 foot per 1″ per 1″, then the given acceleration may be expressed in terms of this unit as follows. (See Arts. 17 and 18 and (*k*) of Introductory Examples.)

80 yards per minute per minute.

$= 80 \times 3$ feet per minute per minute.

$= \frac{80 \times 3}{60}$ feet per second per minute.

$= \frac{80 \times 3}{60 \times 60}$ feet per second per second $= \frac{1}{15}$ feet per 1″ per 1″.

By Art. 12, $s = \frac{1}{2}ft^2 = \frac{1}{2} \times \frac{1}{15} \times 30^2 = 30$ feet.

10. $105 - 75 = 30$ = velocity in feet per second gained in a second. Therefore acceleration = 30 feet per 1″ per 1″. Hence the velocity in the second before the first observed $= 75 - 30 = 45$ feet per 1″, and therefore in this second it passed over a space of 45 feet. Similarly the space described in the previous second $= 45 - 30 = 15$ feet.

11. See preceding Solution. The acceleration $= 33 - 27 = 6$ feet per $1''$ per $1''$. The space described in the 15th second = space in 15 seconds less by space in 14 seconds. Therefore by equation (2) Art. 12,

space in 15th second $= \frac{1}{2} \times 6 \times 15^2 - \frac{1}{2} \times 6 \times 14^2 = \frac{1}{2} \times 6 (15^2 - 14^2) = 87$ feet.

Or: using the principle of Art. 15 (2), since the 15th odd number = 29, the space described in the 15th second is 29 times the space described in the first second $= 29 \times 3 = 87$ feet.

12. We employ the equation of Art. 11 connecting velocity and time, and with − sign since the acceleration is opposite to the initial velocity. Equation (2) Art. 11, $v = u - ft$. Now by the question the final velocity v becomes 0 in 30 seconds $\therefore$

$$0 = u - 15 \times 30 \therefore u = 450 \text{ feet per second.}$$

13. We employ equation connecting velocity and space with − sign $\therefore$ equation (3) Art. 11, $v^2 = u^2 - 2fs$. And by the question the final velocity v becomes 0 when the body has moved through 100 feet, $\therefore 0 = u^2 - 2 \times 8 \times 100 \therefore u^2 = 1600 \therefore u = 40$ feet per second.

14. We employ equations of Art. 14 with + sign. Taking unit of acceleration = 1 metre per second, then (Art. 14) $g = 9\cdot 8$ metres per second.

(*a*) $v = u + gt, \therefore 99 = 50 + 9\cdot 8t \therefore t = 5$ seconds.

(*b*) $s = ut + \frac{1}{2}gt^2 = 50 \times 5 + \frac{1}{2} \times 9\cdot 8 \times 5^2 = 372\cdot 5$ metres.

NOTE.—The above results may also be obtained with the same equations, by taking $g = 32$ feet per $1''$ per $1''$ and by reducing the metres to feet.

15. When the body reaches its greatest height its velocity becomes 0. We employ therefore equation (6) of Art. 14 with − sign, since it connects velocity and space, and the acceleration of gravity is opposite in direction to the initial velocity of projection.

$$v^2 = u^2 - 2gs \therefore 0 = u^2 - 2 \times 9\cdot 8 \times 510\cdot 2 \therefore u = \sqrt{10000} = 100 \text{ metres nearly.}$$

16. We use equation connecting velocity and time without initial velocity.

Art. 14, $v = gt = 9 \cdot 8 \times (20 \times 60) = 11760$ metres per second.

NOTE.—In the two preceding solutions g might have been taken $= 32$ feet per $1''$ per $1''$, and the metres reduced to feet.

17. See solution to Exercise 10.

$137\frac{1}{2} - 112\frac{1}{2} = 25 \therefore$ acceleration $= 25$ feet per $1''$ per $1''$

and $112\frac{1}{2} - 25 = 87\frac{1}{2}$ feet. Also $87\frac{1}{2} - 25 = 62\frac{1}{2}$ feet.

18. The space described in the last two seconds = space in $6''$ − space in $4''$.

$\therefore$ as in solution to Exercise 8, $s = \frac{1}{2}g(6^2 - 4^2) = \frac{1}{2} \times 32 (36 - 16) = 320$ feet.

19. We use equation 5 of Art. 14, with the − sign.

$s = ut - \frac{1}{2}gt^2 \therefore 48 = 64t - 16t^2 \therefore t^2 - 4t = -3.$

Solving this quadratic equation we find $t = 1$ or 3 seconds. Hence, at the end of 1 second the stone will be 48 feet above the ground, and after reaching its greatest height it will, at the end of 3 seconds on its return to the ground, be once more 48 feet above the ground.

20. By question the acceleration $= 8$ feet per $1''$ per $1''$.

Using equation (2) Art. 12, $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 8 \times 5^2 = 100$ feet.

21. We first find the acceleration, using equation (2) Art. 12.

$s = \frac{1}{2}ft^2 \therefore 36 = \frac{1}{2}f \times 3^2 \therefore f = 8$ feet per $1''$ per $1''$.

To find the velocity, we employ equation (1) of Art. 12.

$v = ft = 8 \times 6 = 48$ feet per $1''$.

22. We find the velocity at the end of the first 256 feet, by equation (3) Art. 14. This is the initial velocity with

which the body starts to describe the second 256 feet; and by employing equation (5) Art. 14, we find the time.

$v^2 = 2gs \therefore v = \sqrt{2 \times 32 \times 256} = 128$ feet per second.

$s = ut + \frac{1}{2}gt^2 \therefore 256 = 128t + 16t^2 \therefore t^2 + 8t = 16.$

Solving this quadratic equation we find $t = 1{\cdot}65$ sec.

Or:—Find the time the body takes to fall through 256 feet, and the time it takes to fall through 512 feet: the required time is the difference of these times.

We employ equation (2) Art. 14.

$s = \frac{1}{2}gt^2 \therefore t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2 \times 256}{32}} = 4$ secs. = time to fall through 256 feet.

$s = \frac{1}{2}gt^2 \therefore t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2 \times 512}{32}} = 5 \cdot 65$ secs. = time to fall through 512 feet.

$\therefore$ Time required $= 5 \cdot 65 - 4 = 1 \cdot 65$ secs.

23. Let t = number of seconds previous to the observed second.

Then $t + 1$ = whole number of seconds body has been falling.

Space in the $(t + 1)$ th second = space in $(t + 1)$ seconds − space in t seconds. Employing equation (2) Art. 14, $s = \frac{1}{2}gt^2 \therefore$

Space in $(t + 1)$ th second $= \frac{1}{2}g(t + 1)^2 - \frac{1}{2}gt^2 = \frac{1}{2}g\{(t + 1)^2 - t^2\} = 32t + 16$

But by question the space in the $(t + 1)$ th second $= 176$ feet.

$\therefore 32t + 16 = 176 \therefore t = 5$ seconds.

24. We employ equation (2) Art. 14, since it connects space and time.

$s = \frac{1}{2}gt^2 \therefore t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2 \times 192 \times 3}{32}} = 6$ seconds.

NOTE.—g being taken = 32 feet per 1″ per 1″, the space must be reduced to feet.

25. We find the initial velocity which the body had at B and with which it described the space BC, by employing equation (5) Art. 14. This velocity was that which the body acquired by falling through the space AB, and we find this space by using equation (3) Art. 14.

Let u = velocity which the body acquired by falling from A to B, then this may be regarded as the initial velocity with which the body starts to describe the space BC. To find u:—Art. 14, equation 5,

$s = ut + \frac{1}{2}gt^2 \therefore 144 = 2u + \frac{1}{2} \times 32 \times 2^2$

$\therefore 2u = 80 \therefore u = 40$ feet per second, the velocity at B.

Again, how far must a body fall to acquire a velocity of 40 feet per 1″?

Equation (3) Art. 14, $v^2 = 2gs \therefore 40^2 = 64s \therefore s = 25$ feet.

26. See Art. 15 (1).

27. See Arts. 17 and 18, and Introductory Examples (k).

Let t seconds = the unit of time.

32 feet per second per second

$= \frac{32}{3} \times t \times t$ yards per t'' per t''. Therefore by question

$\frac{32}{3}t^2 = 384 \therefore t = 6$ seconds.

28. Taking unit of acceleration = 1 yard per minute per minute, and employing equation (1) Art. 12.

Then 60 miles per hour $= \frac{60 \times 1760}{60}$ yards per minute

= 1760 yards.

$v = ft \therefore 1760 = f \times 5 \therefore f = 352$.

29. 352 yards per minute per minute

$= 352 \times 3$ feet per minute per minute.

$= \frac{352 \times 3}{60}$ feet per second per minute.

$= \frac{352 \times 3}{60 \times 60} = \frac{22}{75}$ feet per second per second.

NOTE.—See Arts. 17 and 18. It is evident that the number which expresses the measure in feet is 3 times greater than that which expresses it in yards, and the number which expresses the measure in a second is $\frac{1}{60 \times 60}$th part of that in a minute.

30. 30 miles per hour = 30 × 5280 feet per hour = $\frac{30 \times 5280}{60 \times 60}$ feet per second = 44.

31. 32 feet per second per second

$= \frac{32}{3}$ yards per 1″ per 1″

$= \frac{32}{3} \times 60 \times 60$ yards per minute per minute = 38400.

CHAPTER III.

EXAMINATION QUESTIONS.

1. Define Component velocities, and Resultant velocity.
2. What is meant by the Composition of velocities?
3. How is the problem of the composition of velocities divided?
4. How is the resultant of two or more velocities in the same line found?
5. How is the resultant of two uniform velocities in different directions determined?
6. State and prove the Parallelogram of Velocities?
7. Give a proof of this proposition which does not make any assumption regarding the nature or direction of the resultant velocity.
8. Show how the resultant of two uniform velocities may be determined:—(*a*), geometrically; (*b*), numerically.

9. Investigate a general formula by which the resultant velocity may be found when the component velocities and the angle between their directions are given.

10. State and prove the Triangle of velocities.

11. State and prove the Polygon of velocities.

12. State and prove the Parallelopiped of velocities.

13. Define Resolution of velocities. Show how a given velocity may be resolved in a given direction.

14. State the different methods by which the resultant of any number of velocities possessed by a particle may be determined.

15. How may accelerations be compounded and resolved?

INTRODUCTORY EXAMPLES TO CHAP. III.

(*a*) A body has two uniform velocities in the same line and in the same direction, of 15 feet per second, and 120 yards per minute, respectively: what is the resultant velocity?

120 yards per minute $= \dfrac{120 \times 3}{60}$ feet per $1'' = 6$ feet per second.

By Art. 21, resultant = sum of components $= 15 + 6 = 21$ feet per second.

(*b*) A body has two uniform velocities in the same line and in the same direction, of 30 miles per hour and 20 feet per second respectively; and has another velocity of 1800 yards per minute in the same line but in the opposite direction: what is the resultant velocity?

$$30 \text{ miles per hour} = \frac{30 \times 5280}{60 \times 60} = 44 \text{ feet per second.}$$

$$1800 \text{ yards per minute} = \frac{1800 \times 3}{60} = 90 \text{ feet per second.}$$

By Art. 21, Resultant $= 44 + 20 - 90 = -26$ feet per second. Therefore the resultant is a velocity of 26 feet per second in the direction of the last velocity.

(*c*) A body has two uniform velocities of 40 feet and 60 feet per second respectively, in directions inclined at an angle of 60°: what is the resultant velocity?

See Fig. 1 of Art. 26. When the angle AOB or LBM is 60° it is readily seen that $BM = \frac{1}{2}BL$. Therefore using equation (1) Art. 26, $OL^2 = OB^2 + BL^2 + 2OB \cdot BM$ $\therefore OL^2 = 60^2 + 40^2 + 2 \times 60 \times 20 \therefore OL^2 = 7600 \therefore OL = 20\sqrt{19}$ feet per second.

Or :—Employing the general equation of Art. 27 $V^2 = v^2 + v'^2 + 2vv' \cos 60^\circ = 40^2 + 60^2 + 2 \times 40 \times 60 \times \frac{1}{2}$ $\therefore V = \sqrt{7600} = 20\sqrt{19}$ feet per second.

(*d*) A body has uniform velocities of 10 feet and 12 feet per second respectively, inclined at an angle of 135°: what is the resultant?

See Fig. 2, Art. 26. Since $AOB = 135^\circ$, $OBL = 45^\circ$ and $BM = \sqrt{\frac{BL^2}{2}}$

Using equation 2, Art. 26, $OL^2 = OB^2 + BL^2 - 2OB \cdot BM$ $= 12^2 + 10^2 - 2 \times 12 \times \sqrt{\frac{10^2}{2}} \therefore OL^2 = 144 + 100 - 24\sqrt{50} \therefore OL = 8 \cdot 6$ feet per second.

Or :—Using the general equation of Art. 27 $V^2 = v^2 + v'^2 + 2vv' \cos 135^\circ = 10^2 + 12^2 + 2 \times 10 \times 12 \times (-\frac{1}{2}\sqrt{2})$.

$\therefore V^2 = 100 + 144 - 120\sqrt{2} \therefore V = 8 \cdot 6$ feet per second.

(*e*) A body has two uniform velocities of 40 feet per second and 600 yards per minute respectively, in directions inclined to each other at a right angle: what is the resultant?

By Art. 26, $\text{Resultant}^2 = 40^2 + 60^2 \therefore \text{Resultant} = \sqrt{2500} = 50$ feet per 1″.

Or :—By Art. 27, $V^2 = v^2 + v'^2 + 2vv' \cos 90^\circ = 40^2 + 30^2 + 2 \times 40 \times 30 \times 0 = 40^2 + 30^2 \therefore V = 50$ feet per second.

(*f*) A body has two uniform velocities of 50 metres and 60 metres per second respectively, in directions inclined to each at a right angle: what is the resultant?

As in last question $V^2 = v^2 + v'^2 = 50^2 + 60^2 \therefore V = 10\sqrt{61}$ metres per second.

(*g*) A velocity of 100 feet per second is inclined to the horizon at an angle of 30°: what is the value of this velocity in a horizontal direction?

Art. 31, Resolved velocity $= 100 \cos 30^\circ = 100\frac{\sqrt{3}}{2}$

$= 50\sqrt{3}$ feet per second.

SOLUTIONS OF EXERCISES TO CHAPTER III. PAGE 25.

1. If the velocities be all in the same line, then (Art. 21) if they kept the body at rest, two of the velocities would be equal in amount and opposite in direction to the third; but this cannot be the case with the given velocities.

If the velocities be not all in the same line, then (Art. 28) if they kept the body at rest, a triangle could be formed, having its sides proportional to the velocities and parallel respectively to their directions. But since one of the velocities is greater than the sum of the other two, no triangle could be formed having its sides proportional to the given velocities. Hence the body cannot remain at rest in any case.

2. 3 miles per hour $= \frac{3 \times 5280}{60 \times 60} = 4 \cdot 4$ feet per second.

180 yards per minute $= \frac{180 \times 3}{60} = 9$ feet per second.

Then Art. 21, Resultant $= 9 - 4 \cdot 4 = 4 \cdot 6$ feet per second.

3. The angle between the directions is a right angle and therefore by Art. 26, Resultant$^2 = 8^2 + 6^2$ $\therefore$ R $= \sqrt{8^2 + 6^2}$ = 10 feet feet per second.

Or, using equation of Art 27,

$V^2 = v^2 + v'^2 + 2vv' \cos 90^\circ$ and since $\cos 90^\circ = 0$

$\therefore V^2 = v^2 + v'^2 = 8^2 + 6^2$ $\therefore V = \sqrt{100} = 10$ feet per second.

4. 100 yards per minute $= \frac{100 \times 3}{60}$ feet per second $= 5$ feet per second.

Then, Art. 21, resultant velocity $= 20 + 25 - 5 = 40$ feet per second.

To find the space described with this uniform velocity in 10 seconds, we use the equation of Art. 6:—
$s = ut = 40 \times 10 = 400$ feet.

5. By Art. 31, the component in a horizontal direction $= 100 \cos 60^\circ = 100 \times \frac{1}{2} = 50$ feet per second.

The component in a vertical direction $= 100 \sin 60^\circ =$
$100 \times \frac{\sqrt{3}}{2} = 50\sqrt{3}$ feet per 1″

6. By Art. 31, the velocity in a vertical direction $= 500$ ($\sin 60^\circ$ or $\cos 30^\circ$) $= 500 \times \frac{\sqrt{3}}{2} = 250\sqrt{3}$ feet per second.

7. Each velocity has its full effect independent of the other. The time taken to cross will be the same whether the water is flowing or at rest, that is if there were no current, the time taken to cross with the given rate of rowing would be the same as the time that is taken to describe the actual path with the resultant velocity compounded of the rowing and the current of the stream.

Hence since the width of the stream is 440 yards, and the boat is rowed at the rate of 44 yards per minute, $\frac{440}{44} = 10$ minutes, the time taken to cross. And since during this time the boat is carried with the stream at the rate of 10 yards per minute, it will be carried down the stream $10 \times 10 = 100$ yards.

See Exam. 1, page 75.

8. By Art. 21, the resultant velocity up the stream $= 10 - 2 = 8$ miles per hour, and the velocity down the stream $= 10 + 2 = 12$ miles per hour.

9. Since the rate down is increased by the rate of the stream, and that up is diminished by the rate of the stream, it follows that the difference of the rates down and up is twice the rate of the stream. Hence $7\frac{1}{2} - 6\frac{1}{2} = 1$ mile per half-hour $= 2$ miles per hour $=$ twice rate of stream.

Therefore the rate of the stream $= 1$ mile per hour.

10. (*The student will draw the diagram.*) Draw AC respresenting the velocity southwards, and AB at right angles to AC representing the velocity of the ball to the west; complete the rectangle ABDC and join AD. Then Art. 26 or 27, $AD^2 = AB^2 + AC^2 \therefore V^2 = 5^2 + (5\sqrt{3})^2 \therefore V = \sqrt{25 + 75} = 10$ feet per second.

Again in the triangle DAC, by Trigonometry, $\frac{DC}{AD} = \sin A \therefore \sin A = \frac{5}{10} = \frac{1}{2} = \sin 30^\circ \therefore A = 30^\circ$.

11. (*The student will draw the diagram.*) Let AD represent the resultant velocity $= 50$ feet. From A draw $AB = 25$ feet, and making an angle of 60° with AD. Join BD. Then the triangle ABD is evidently half of an equilateral triangle, and the angle ABD is a right angle. Therefore $BD^2 = AD^2 - AB^2 = 50^2 - 25^2 \therefore BD = 25\sqrt{3}$. Complete the rectangle ABDC; AC = the other component = BD $= 25\sqrt{3}$ feet per second.

12. The distances travelled in half an hour will be 6 miles and 5 miles respectively, and these distances will form the sides of a right-angled triangle whose hypotenuse will be the distance apart. Therefore, by Geometry, Distance $= \sqrt{6^2 + 5^2} = \sqrt{61} = 7 \cdot 8$ miles.

13. The line joining their positions at any time forms one side of an equilateral triangle whose other sides are the distances travelled along the roads. Hence the distance apart is equal to the distance travelled by each $= 1\frac{1}{2}$ mile.

14. 15 miles per hour $= \frac{15 \times 5280}{60 \times 60} = 22$ feet per second.

By Art. 26 or 27, $V = \sqrt{22^2 + 10^2} = 24 \cdot 16$ feet per second.

15. We may proceed either according to the method of Art. 26 or of Art. 27.

Employing the equation of Art. 27, we have

(a) $V^2 = 100^2 + 100^2 + 2 \times 100 \times 100 \times \cos 30^\circ$
$= 10000 + 10000 + 20000\frac{\sqrt{3}}{2} \therefore V = 193 \cdot 2$ feet per 1″

(b) $V^2 = 100^2 + 100^2 + 2 \times 100 \times 100 \times \cos 120^\circ$
$= 10000 + 10000 + 20000\left(-\frac{1}{2}\right) \therefore V = 100$ feet per 1″.

16. 15 miles per hour $= \frac{15 \times 5280}{60}$ feet per minute $= 1320$ feet per minute.

Therefore the velocities are as 25 : 1320 or as 5 : 264.

17. 1 mile in 10 minutes $= \frac{5280}{10 \times 60} = 8{\cdot}8$ feet per second.

100 yards in 10 seconds $= \frac{100 \times 3}{10} = 30$ feet per second.

Therefore ratio $= 8 \cdot 8 : 30$ or as 22 : 75.

18. See Art. 32. Resolving along the direction of the velocity 10 and at right angles, and calling the sum of the resolved parts in the first direction X, and in the second Y, we have

$X = 10 - 8 \cos 60^\circ - 8 \cos 60^\circ = 10 - 8 \times \frac{1}{2} - 8 \times \frac{1}{2} = 2$

$Y = 0 + 8 \cos 30^\circ - 8 \cos 30^\circ = 0.$

Since the sum of the resolved velocities in the latter direction vanishes, the resultant is the sum in the direction of the velocity 10, equal 2 feet per second.

19. Resolving in the east and west direction, and then in the north and south; and calling the sum of the resolved velocities in the former direction X, and in the latter Y, we have

$X = 60 - 20 = 40$. And $Y = 30$. Therefore by Art. 32, Resultant $= \sqrt{X^2 + Y^2} = \sqrt{40^2 + 30^2} = 50$ feet per second.

20. By Art. 26 or Art. 27,

$$V^2 = \sqrt{10^2 + 8\cdot8^2} = \sqrt{177\cdot44} = 13\cdot3 \text{ feet per second.}$$

21. Since each of the velocities is in a direction at right angles to the other two, we employ the equation of Art. 30.

$$\therefore V = \sqrt{22^2 + 4\cdot4^2 + 8^2} = 23\cdot8 \text{ feet per second.}$$

In the above equation 22 feet per $1''$ = 15 miles per hour, and 4·4 feet per $1'' = 3$ miles per hour.

22. We employ equation of Art. 27 to find the resultant velocity.

$\therefore V^2 = 12^2 + 3^2 + 2 \times 12 \times 3 \times \cos 45^\circ = 144 + 9 + 36\sqrt{2}$ $= 14\cdot28$ miles per hour.

To find the distance traversed with this uniform velocity, we employ the equation of Art. 6. $s = ut = 14\cdot28 \times 4 = 57\cdot1$ miles.

23. By Art. 30 the Resultant acceleration

$= \sqrt{20^2 + 30^2 + (20\sqrt{3})^2} = 50$ feet per $1''$ per $1''$.

To find the space described with this acceleration we employ equation (2) Art. 12. $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 50 \times 10^2$ $= 2500$ feet.

24. By equation of Art. 27, $V^2 = 20^2 + 50^2 + 2 \times 20 \times 50 \times \cos 45^\circ = 20^2 + 50^2 + 2000 \times \frac{1}{2}\sqrt{2} \therefore V = 65\cdot68$ metres per second.

By Art. 6, $s = ut = 65\cdot68 \times 2 = 131\cdot3$ metres.

25. Resolving horizontally in each case, we have by Art. 31

(1) $50 \cos 45^\circ = 50 \times \frac{1}{2}\sqrt{2} = 25\sqrt{2}$ feet per sec. per sec.

(2) $100 \cos 45^\circ = 100 \times \frac{1}{2}\sqrt{2} = 50\sqrt{2}$ feet per sec. per sec.

(3) $100\sqrt{2} \cos 45^\circ = 100\sqrt{2} \times \frac{1}{2}\sqrt{2} = 100$ feet per sec. per sec.

26. We obtain the resultant velocity by Art. 26 or Art. 27.

And since 15 miles per hour = 22 feet per second $\therefore$

(1) $V^2 = 22^2 + 10^2 \therefore V = 24 \cdot 16$ feet per second.

(2) By Art. 6, $s = ut = 24 \cdot 16 \times 3 = 72 \cdot 48$ feet.

27. By Art. 27, $V^2 = v^2 + v' + 2vv' \cos 120^\circ = 22^2 + 10^2 + 2 \times 22 \times 10 \times (-\frac{1}{2}) \therefore V = \sqrt{364} = 19 \cdot 08$ feet per sec.

28. By Art. 27, $V^2 = 10^2 + 12^2 + 2 \times 10 \times 12 \times \cdot 3 \therefore V = \sqrt{316} = 17 \cdot 7$ feet per 1″.

29. (1) By Art. 27,

$V^2 = 20^2 + 30^2 + 2 \times 20 \times 30 \times \frac{1}{2} \therefore V = \sqrt{1600} = 40$ feet per 1″.

(2) By Art. 12, $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 40 \times 10^2 = 2000$ feet.

30. (*The student will draw the diagram.*) Let O be the body. Draw OB horizontal and = 10; draw OA = OB making an angle of 30° with OB above the horizontal line; draw OC = OB making an angle of 60° below the horizontal; and draw OD = OB vertically downwards. These lines represent in magnitude and direction the given velocities.

(1) Resolve vertically and horizontally, as in Art. 32. Then calling the sum of the resolved velocities in a vertical direction X, and the sum of those in a horizontal direction Y, we have

$X = 10 \cos 60^\circ + 0 - 10 \cos 30^\circ - 10 = 5 \times \frac{1}{2} - 5\sqrt{3} - 10$
$= -5 - 5\sqrt{3}$.

$Y = 10 \cos 30^\circ + 10 + 10 \cos 60^\circ + 0 = 5\sqrt{3} + 10 + 10 \times \frac{1}{2}$
$= 15 + 5\sqrt{3}$.

$\therefore R = \sqrt{X^2 + Y^2} = \sqrt{746 \cdot 39} = 27 \cdot 32$ feet per sec.

(2) By Art. 6, $s = ut = 27 \cdot 32 \times 5 = 136 \cdot 6$ feet.

CHAPTER IV.

EXAMINATION QUESTIONS.

1. State Newton's First Law of Motion.

2. Define Matter and Force, and show how these definitions are obtained.

3. What is Inertia? Give illustrations of inertia.

4. When a body is at rest, or in uniform motion in a straight line, what may be inferred with regard to the action of forces on the body?

5. Mention different kinds of forces.

6. How may forces be represented?

7. How may forces be measured?

8. If a force act constantly on a body which is perfectly free to move, what kind of velocity is produced? If at any instant the force cease to act, and no other force acts on the body, with what kind of velocity will the body move? Give the reasons for your answers.

9. When a body moves upon a smooth inclined plane, how may the relations between space, time, and acceleration be determined?

10. Show that the velocity acquired by a body in falling down an inclined plane is equal to what it would acquire in falling freely through the height of the plane.

11. If chords be drawn from the highest point of a vertical circle, show that the time taken by a body to fall down any chord is the same as that taken to fall through any other chord.

12. Determine the line of quickest descent from a given point to a given line in the same vertical plane.

13. When a body is projected vertically upwards in a vacuum show:—

(1) The time of ascent is equal to the time of descent.

(2) The velocity when it returns to the point of projection is equal to the velocity with which it was projected, but is opposite in direction.

(3) The velocity at any point in its ascent is equal and opposite to its velocity at the same point in its descent.

14. How are the results stated in the last question modified when the body moves in the air instead of in a vacuum?

INTRODUCTORY EXAMPLES.

(*a*) A plane is inclined to the horizon at an angle of 30°, and a body falls from rest down the plane: (1) What velocity will it acquire in 6 seconds; (2) What space will it describe in that time?

See Art. 43. We employ the equations of Art. 14, writing $g \sin i$ for g $\therefore$

(1) $v = g \sin 30^\circ\, t = 32 \times \frac{1}{2} \times 6 = 96$ feet per second.

(2) $s = \frac{1}{2} g \sin 30^\circ\, t^2 = \frac{1}{2} \times 32 \times \frac{1}{2} \times 6^2 = 288$ feet.

(*b*) If in the preceding question the body had been thrown down with a velocity of 50 feet per second, (1) What velocity would it acquire in 3 seconds, (2) What space would it describe in 5 seconds?

We employ equations (4) and (5) of Art. 14, writing $g \sin i$ for g.

(1) $v = u + g \sin 30^\circ\, t = 50 + 32 \times \frac{1}{2} \times 3 = 98$ feet per second.

(2) $s = ut + \frac{1}{2} g \sin 30^\circ\, t^2 = 50 \times 5 + \frac{1}{2} \times 32 \times \frac{1}{2} \times 5^2 = 450$ feet.

(*c*) A body is thrown vertically upwards from the ground with a velocity of 256 feet per second: (1) What velocity will it have in 4 seconds? (2) How high will it be above the ground in 2 seconds?

We use equations (1) and (2) of Art. 47.

(1) $v = u - gt = 256 - 32 \times 4 = 128$ feet per second.

(2) $s = ut - \frac{1}{2} gt^2 = 256 \times 2 - \frac{1}{2} \times 32 \times 2^2 = 448$ feet.

(*d*) In the foregoing case find the velocity in 8 seconds, and the height above the ground in 16 seconds.

(1) $v = u - gt = 256 - 32 \times 8 = 0$ feet per 1″. That is, the body has reached its greatest height and has come to rest.

(2) $s = ut - \frac{1}{2}gt^2 = 256 \times 16 - \frac{1}{2} \times 32 \times 16^2 = 0$ feet. That is, the body has returned to the ground.

SOLUTIONS OF EXERCISES TO CHAP. IV. Page 37.

1. (1) Art. 14, equation (4), $v = u - gt \therefore t = \frac{u - v}{g} = \frac{5g - 2g}{g}$ $= 3$ seconds.

(2) Art. 14, equation (5), $s = ut - \frac{1}{2}gt^2 \therefore 8g = 5gt - \frac{1}{2}gt^2 \therefore t^2 - 10t = -16$. Solving for t we find $t = 2$ seconds or 8 seconds.

Hence in 2 seconds the body will rise to the height $8g$. It will then go up to its greatest height, will return again, and at the end of 8 seconds will once more be at the height of $8g$ above the ground.

2. We employ equation (5) Art. 14 with $-$ sign :—

$s = ut - \frac{1}{2}gt^2 \therefore$

(*a*) Height $= 96 \times 2 - \frac{1}{2} \times 32 \times 2^2 = 128$ feet.
(*b*) „ $= 96 \times 5 - \frac{1}{2} \times 32 \times 5^2 = 80$ feet.
(*c*) „ $= 96 \times 6 - \frac{1}{2} \times 32 \times 6^2 = 0$ feet.
(*d*) „ $= 96 \times 10 - \frac{1}{2} \times 32 \times 10^2 = -640$ feet.
(*e*) „ $= 96 \times 12 - \frac{1}{2} \times 32 \times 12^2 = -1152$ feet.

The answer to (*c*) shows that the stone has returned in 6 seconds to the point of projection. The $-$ sign in the answers to (*d*) and (*e*) shows that the stone has passed *below* the point of projection.

3. Let $t =$ the time taken by each body till they meet. Then the space described by each is given by equation (5) of Art. 14, taking the $-$ sign in the case of the body projected upwards, and the $+$ sign for that downwards $\therefore$

Space described by 1st body $= ut - \frac{1}{2}gt^2 = 64t - 16t^2$
Space described by 2nd body $= ut + \frac{1}{2}gt^2 = 64t + 16t^2$.

But the sum of the space = height of tower $\therefore$ adding the equations and putting the sum = the height, we have

$$128t = 128 \therefore t = 1 \text{ second.}$$

Substituting the value of t in each of the above equations, we have the space described by the first $= 64 \times 1 - 16 \times 1^2 = 48$ feet, and by the second $= 80$ feet. Therefore the stones will meet at a point 48 feet from the base, and 80 feet from the top.

4. Let t = the number of seconds that elapse till the stones meet. We employ equation (5) of Art. 14 with − sign to obtain the space described by the body projected upwards, and equation (2) Art. 14 for that allowed to fall from rest. Hence as before

Space described by 1st $= ut - \frac{1}{2}gt^2 = 64t - 16t^2$.

Space described by 2nd $= \frac{1}{2}gt^2 = 16t^2$.

The sum of these spaces = height of tower $\therefore$ adding the equations,

$$64t = 128 \therefore t = 2 \text{ seconds.}$$

Substituting this value of t in each of the equations, we find the space described by each = 64 feet $\therefore$ the bodies meet half way.

5. (1) When the body reaches its greatest height its velocity becomes 0 $\therefore$ problem (1) may be expressed as follows:—Find the height to which the body will reach when its velocity becomes 0. We therefore employ equation (6) of Art. 14 with − sign, since it is the equation connecting velocity and space, and the initial velocity is opposite in direction to the acceleration of gravity $\therefore v^2 = u^2 - 2gs$. And as the velocity becomes 0 when the greatest height is reached, we have $0 = 120^2 - 2 \times 32 \times s \therefore s = \dfrac{120^2}{2 \times 32} = 225$ feet.

(2) We employ equation (5) Art. 14, with − sign.

$s = ut - \frac{1}{2}gt^2 = 120 \times 5 - \frac{1}{2} \times 32 \times 5^2 = 200$ feet.

(3) $s = ut - \frac{1}{2}gt^2 = 120 \times 10 - \frac{1}{2} \times 32 \times 10 = -400$ feet.

The answer to (3) shows that in 10 seconds, the body has come back and has passed 400 feet *below* the point of projection.

6. From Art. 47, the time of flight $= \frac{2u}{g} = \frac{2 \times 120}{32} = 7\frac{1}{2}$ seconds.

7. We have first to determine the velocity of projection. And since by the question it *just* reaches the top of the tower, it follows that its velocity becomes 0 when it reaches the height of 256 feet. Hence this part of the question may be expressed as follows:—With what velocity must a body be projected so that when its velocity becomes 0 it will have reached the height of 256 feet? We employ equation (6), Art. 14. with - sign, since it is the equation connecting space and velocity $\therefore$

$v^2 = u^2 - 2gs \therefore 0 = u^2 - 2 \times 32 \times 256 \therefore u = 128$ feet per second.

The first part of the question may now be stated as follows:—A body is projected vertically upwards with a velocity of 128 feet per second: when will it reach the height of 128 feet? We employ equation (5), Art. 14, connecting space and time.

(1) $s = ut - \frac{1}{2}gt^2 \therefore 128 = 128t - 16t^2 \therefore t^2 - 8t = -8$.

$\therefore t = 1 \cdot 17$ sec. and $6 \cdot 82$ sec. That is in $1 \cdot 17$ sec. it passes the middle point of the tower going up, and in $6 \cdot 82$ seconds it passes the middle point coming down.

(2) Since the height of the tower is the greatest height to which the body reaches, therefore Art. 47,

Time of ascent $= \frac{u}{g} = \frac{128}{32} = 4$ seconds.

8. To find the velocity at the middle point of the tower going up, we use equation (6), Art. 14, connecting velocity and space.

$v^2 = u^2 - 2gs = 128^2 - 2 \times 32 \times 128 \therefore v = 64\sqrt{2} = 90 \cdot 5$ feet per second.

And from Art. 47 the velocity at the middle point coming down is the same as that at the same point going up, but is of opposite sign.

9. See Art. 43. We employ equation (1) of Art. 14, writing $g \sin i$ for g.

$$v = g \sin i \; t = 32 \times \frac{1}{10} \times 4 = 12 \cdot 8 \text{ feet per second.}$$

10. We use equation (2), Art. 14, writing $g \sin i$ for g.
$s = \frac{1}{2} g \sin 30^\circ \; t^2 = \frac{1}{2} \times 32 \times \frac{1}{2} \times 5^2 = 200$ feet.

11. When the body reaches the top of the plane its velocity becomes 0, hence, as in solution to Exercise 7, the question may be expressed: What is the length of the plane rising 1 in 16, so that when a body is thrown up the plane with a velocity of 128 feet per 1″, its velocity may become 0 when it reaches the top? We employ equation (6) of Art. 14 with − sign, but writing $g \sin i$ for g ∴

$$v^2 = u^2 - 2g \sin i \; s \therefore 0 = 128^2 - 2 \times 32 \times \frac{1}{16} \times s \therefore s = \frac{128^2}{2 \times 32 \times \frac{1}{16}} = 4096 \text{ feet.}$$

12. Art. 14, equation (2), writing $g \sin i$ for g.

$$s = \frac{1}{2} g \sin i \; t^2 \therefore 128 = \frac{1}{2} \times 32 \times \frac{1}{2} \times t^2 \therefore t = 4 \text{ seconds.}$$

13. (*a*) We use equation connecting velocity and time, writing $g \sin i$ for g.

$$\therefore v = u - g \sin i \; t = 50 - 32 \times \frac{1}{10} \times 5 = 34 \text{ feet per second.}$$

(*b*) This part of the question is equivalent to:—In what time will the velocity be 0? We therefore use the same equation, putting $v = 0$.

$$v = u - g \sin i \; t \therefore 0 = 50 - 32 \times \frac{1}{10} \times t \therefore t = \frac{50}{32 \times \frac{1}{10}} = 15\frac{5}{8} \text{ seconds.}$$

14. See solutions to Exercises 7 and 11. We use equation (6) of Art. 14, writing $g \sin i$ for g. $v^2 = u^2 - 2g \sin i\, s \therefore 0 = u - 2 \times 32 \times \frac{1}{64} \times 225 \therefore u^2 = 2 \times 32 \times \frac{1}{64} \times 225 \therefore u = 15$ feet per second.

15. We employ equations (4) and (5) of Art. 14, writing $g \sin i$ for g, and with + sign, since the initial velocity is in the same direction as the acceleration of gravity.

(1), $v = u + g \sin i\, t = 50 + 32 \times \frac{\sqrt{3}}{2} \times 3 = (50 + 48\sqrt{3})$ feet.

(2), $s = ut + \frac{1}{2}g \sin i\, t^2 = 50 \times 3 + 16 \times \frac{\sqrt{3}}{2} \times 9 = (150 + 72\sqrt{3})$ feet.

16. Here since there is no initial velocity, we use equation (2) Art. 14, substituting $g \sin i$ for g.

$$s = \frac{1}{2}g \sin i\, t^2 = \frac{1}{2} \times 32 \times \frac{1}{100} \times (5 \times 60)^2 = 14400 \text{ feet.}$$

17. We now employ equation (5) since there is an initial velocity. And since 15 miles per hour = 22 feet per second, we have

$$s = ut + \frac{1}{2}g \sin i\, t^2 = 22 \times 60 + \frac{1}{2} \times 32 \times \frac{1}{100} \times 60^2 = 1896 \text{ feet.}$$

18. Let t = No. of seconds taken by the 1st stone to overtake the 2nd.

Then $t - 2$ = No. of seconds 2nd stone has been in motion.

Employing equation (2) of Art. 14 we find the space described by each stone, and by the question the difference of these spaces = 192 feet.

$\therefore \frac{1}{2}gt^2 - \frac{1}{2}g\,(t-2)^2 = 192$

$\therefore 16t^2 - 16t^2 + 64t - 64 = 192$

$\therefore 64t = 256 \therefore t = 4$ seconds, the time taken by 1st stone to overtake 2nd. And $t - 2 = 4 - 2 = 2$ seconds, the time the 2nd has been in motion.

The space described by the first stone in 4 seconds is found from equation (2), Art. 14.

$s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times 4^2 = 256$ feet, the distance from the top of the precipice.

19. Let t = No. of seconds taken by the 1st before being overtaken.

Then $t - 2$ = No. of seconds taken by the 2nd to overtake the first.

We employ equation (5) of Art. 14 with + sign to find the space in each case, and as these spaces are equal we have

$$20t + \tfrac{1}{2} \times 32 \times t^2 = 100\,(t - 2) + \tfrac{1}{2} \times 32 \times (t - 2)^2$$

$$\therefore 20t + 16t^2 = 100t - 200 + 16t^2 - 64t + 64$$

$\therefore 16t = 136 \therefore t = 8\tfrac{1}{2}$ seconds, the time that elapses from the starting of the 1st stone.

20. The solution is the same as in the last exercise only that we use the − sign in equation (5), Art. 14.

Making the same supposition as in last solution, then,
Space described by 1st stone $= ut - \tfrac{1}{2}gt^2 = 120t - 16t^2$, and
Space described by 2nd stone $= u\,(t - 2) - \tfrac{1}{2}g\,(t - 2)^2$
$= 200\,(t - 2) - 16\,(t - 2)^2 \therefore$

Since spaces are equal, $120t - 16t^2 = 200\,(t - 2) - 16\,(t - 2)^2 \therefore$

$t = 3\tfrac{2}{9}$ seconds, the time that elapses after the starting of the first stone.

21. We may calculate the vertical diameter, and then find by equation (2) of Art. 14 the time taken to fall freely through this height. This, by Art. 44, will be the time taken to fall down the chord.

Or, we may proceed as follows. Since the chord makes an angle of 45° with the vertical diameter, the inclination of the chord to the horizon is also 45°. Employing therefore equation (2), Art. 14, and writing g sin i for g (Art. 43), we have

$$s = \tfrac{1}{2}g \sin i\; t^2 \therefore 32 = \tfrac{1}{2} \times 32 \times \tfrac{1}{2}\sqrt{2} \times t^2 \therefore$$

$$t = \sqrt{\frac{2}{\frac{1}{2}\sqrt{2}}} = \sqrt{\frac{4}{\sqrt{2}}} = \sqrt{\frac{4\sqrt{2}}{2}} = \sqrt{2\sqrt{2}} \text{ seconds.}$$

22. We may use either of the methods of last solution. The inclination of the chord to the horizon is in this case 60°. Therefore, employing the same equation as in last solution, we have

$$s = \tfrac{1}{2}g \sin 60^\circ\, t^2 \therefore 24 = \tfrac{1}{2} \times 32 \times \frac{\sqrt{3}}{2} \times t^2 \therefore$$

$$t = \sqrt{\frac{3}{\sqrt{3}}} = \sqrt{\sqrt{3}} = \sqrt[4]{3} \text{ seconds.}$$

23. Since the body receives an acceleration of 2 feet per second in one direction, then in 5 seconds its velocity is given by equation (1), Art. 12, $v = ft = 2 \times 5 = 10$ feet per second. Since the velocity in the other direction is uniform it will always possess this uniform velocity in that direction. Hence at the end of 5 seconds the body possesses in each direction a velocity of 10 feet per second; and its resultant velocity is obtained from the equation of Art. 27,

$$V^2 = v^2 + v'^2 + 2vv' \cos 60^\circ = 10^2 + 10^2 + 2 \times 10 \times 10 \times \tfrac{1}{2} \therefore$$
$$V = 10\sqrt{3} = 17 \cdot 32 \text{ feet.}$$

24. The horizontal velocity will always be the same (see Art. 42). The vertical velocity in 5 seconds is obtained from equation (1), Art. 14.

$v = gt = 32 \times 5 = 160$ feet per second. The resultant velocity is obtained from equation of Art. 27, the last term of the equation vanishing since $\cos 90^\circ = 0$.

$V^2 = 100^2 + 160^2 \therefore V = 188 \cdot 6$ feet per second.

The distance of the body at the end of the 5 seconds from the point of projection is evidently the hypotenuse of a right-angled triangle of which the sides are the spaces that would respectively be described in 5 seconds with each velocity. Now with the uniform velocity the space described would be, by Art. 6, $= ut = 100 \times 5 = 500$ feet. And with the accelerated velocity, by equation (2), Art. 12, the space $= \tfrac{1}{2}ft^2 = \tfrac{1}{2} \times 32 \times 5^2 = 400$. Hence the distance from the point of projection $= \sqrt{500^2 + 400^2} = 640 \cdot 9$ feet.

25. As in last question the uniform velocity at end of 5 seconds = 48 feet. The velocity produced by gravity in 5 seconds is found from equation (1), Art. 14. $v = gt = 32 \times 3 = 96$ feet per second. The angle between the two directions $= 90^\circ + 30^\circ = 120^\circ$ ∴ (Art. 27) $V^2 = v^2 + v'^2 + 2vv' \cos 120^\circ$ ∴ $V^2 = 48^2 + 96^2 + 2 \times 48 \times 96 \times (-\frac{1}{2}) = 6912$ ∴ $V = 83 \cdot 1$ feet per second.

26. The uniform velocity, by Art. 6, $= \frac{s}{t} = \frac{512}{4} = 128$ feet per second.

The acceleration by equation (2), Art. 12, $= \frac{2s}{t^2} = \frac{2 \times 256}{16}$ $= 32$ feet per 1″ per 1″.

The uniform velocity remains constant (Art. 42). The accelerated velocity in 4 seconds is found by equation (1), Art. 12.

$v = ft = 32 \times 4 = 128$ feet per second.

The resultant velocity is found from Art. 27.

$$V^2 = v^2 + v'^2 + 2vv' \cos 120^\circ = 128^2 + 128^2 + 2 \times 128 \times 128 \times (-\tfrac{1}{2})$$

∴ $V = 128$ feet per sec.

27. The stone partakes of the upward motion of the balloon so that if it were merely dropped it would have an upward velocity of 50 feet per second, but being thrown up its initial upward velocity $= 50 + 30 = 80$ feet per second. Its distance from the point of projection in 10 seconds is found by using equation (5), Art. 14, with − sign, since it is the equation connecting space and time.

$s = ut - \frac{1}{2}gt^2 = 80 \times 10 - 16 \times 100 = -800$ feet.

That is in 10 seconds the stone was 800 feet *below* the point of projection; and as by the question the stone then reached the ground, it follows that the ground was 800 feet under the balloon at the instant the stone was projected.

28. (1) Since the stone takes 10 seconds to reach the ground, and during this time the balloon ascends uniformly at the rate of 50 feet per second, it rises through a height of $10 \times 50 = 500$ feet above the point where the stone was projected; and as this point by last question was 800 feet above the ground, therefore the balloon in 10 seconds is at a height above the ground of $500 + 800 = 1300$ feet.

The greatest height above the point of projection that the stone reaches is found as in Art. 47, or as in solution to Exercise 7, by employing equation (6), Art. 14, with – sign.

$v^2 = u^2 - 2gs$. And as the velocity becomes 0 when the greatest height is reached $\therefore 0 = 80^2 - 2 \times 32 \times s \therefore s = 100$ feet, that is the stone rises to a height of 100 feet above the point of projection, and as this point was 800 feet above the ground, the greatest height above the ground that the stone attains $= 100 + 800 = 900$ feet.

29. By Art. 21, the resultant of the two velocities given to the stone by the motion of the balloon and by the upward throw $= 0$ $\therefore$ as there is no initial velocity we employ equation (2) of Art. 14, to find the space described in 5 seconds; this will be the height of the balloon above the ground at the instant of projection.

$\therefore s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times 5^2 = 400$ feet.

30. As the effect of each velocity is independent of the other (see Art. 42), the time taken by the stone to reach the ground is the same as if it were merely dropped from the top of the tower, and the horizontal distance described is the same as if gravity did not act on the stone. Hence to find the height we employ equation (2), Art. 14, and to find the horizontal distance, we employ equation of Art 6 $\therefore$

(1) $s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times \left(\frac{5}{2}\right)^2 = 100$ feet.

(2) Space $= ut = 100 \times 2\frac{1}{2} = 250$ feet.

(3) The distance from the point of projection to the point where the stone strikes the ground is the hypotenuse of a right-angled triangle, the perpendicular side being 100 feet and the base 250 $\therefore$

$$\text{Distance} = \sqrt{100^2 + 250^2} = \sqrt{72500} = 269 \cdot 2 \text{ feet.}$$

31. See Art. 36. In the first case owing to inertia the man's body does not at once take up the motion of the boat, he is therefore left behind, and falls backward. In the second case owing to inertia his body tends to go on with the motion it has acquired and he therefore falls forward.

32. See Art. 42. Since the screw has acquired the onward motion of the carriage, and while it is falling it and the carriage are moving forward with the same velocity, it follows that *relative to the carriage* it has no onward motion, and therefore to a person in the carriage it appears to fall vertically just as if during its fall the carriage were at rest. And since the effects of the two velocities are independent of each other, we find the distances in a horizontal and in a vertical direction, by employing the same equations as in solution to Exercise 30.

(2) Since the screw has an onward velocity of 30 miles per hour, or 44 feet per sec., therefore in 1 sec. the space described is (Art. 6) $= ut = 44 \times 1 = 44$ feet $=$ horizontal distance. And (3), by Art. 14, $s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times 1^2 = 16$ feet.

33. The ball is thrown vertically upwards, and having acquired the horizontal velocity of the rider moves during its time of flight as far in a horizontal direction as the rider does, and therefore returns to his hand.

34. See Art. 42. The height and the time of flight will be the same as though there were no horizontal velocity, and the range or horizontal space described will be the same as though there were no vertical velocity.

See Art. 47, and solution to Exercise 7.

(1) To find the greatest height, or to find the height when the velocity becomes 0. $v^2 = u^2 - 2gs$ ∴

$0 = 80^2 - 2 \times 32 \times s$ ∴ $s = 100$ feet.

(2) Art. 47, Time of flight $= \frac{2u}{g} = \frac{2 \times 80}{32} = 5$ seconds.

(3) Art. 6, Range $= ut = 50 \times 5 = 250$ feet.

35. In 10 seconds the height ascended with a uniform velocity of 16 feet per second $= 10 \times 16 = 160$ feet. The stone being now let fall from the car, it possesses an upward initial velocity equal to that of the balloon = 16 feet per sec. The question "when will it reach the ground?" is equivalent to the following: When will it be 160 feet *below* the point of projection?

∴ We employ equation (5) of Art. 14 with − sign.

$s = ut - \frac{1}{2} gt^2$. And since the space $s = 160$ feet *below* the point of projection, we make it negative in the equation ∴ -160 feet $= 16t - 16t^2$ ∴ $t^2 - t = 10$ ∴ $t = 3{\cdot}7$ secs.

36. In 2 seconds the height reached by the lift is found by equation (2), Art. 12, $s = \frac{1}{2}ft^2 = \frac{1}{2} \times 16 \times 4 = 32$ feet.

The velocity at that instant is found by equation (1), Art. 12, $v = ft = 16 \times 2 = 32$ feet per second. This is the upward initial velocity of the stone when allowed to fall from the lift. Hence the question is equivalent to the following: A stone is thrown upwards with a velocity of 32 feet per second, when will it be 32 *below* the point of projection? We proceed exactly as in last solution.

$s = ut - \frac{1}{2}gt^2$ ∴ $-32 = 32t - 16t^2$ ∴ $t^2 - 2t = 2$ ∴ $t = 2{\cdot}73$ seconds.

37. See last solution. Space ascended in 10 seconds $= \frac{1}{2}ft^2 = \frac{1}{2} \times 16 \times 100 = 800$ feet.

Initial upward velocity of stone $= ft = 16 \times 10 = 160$ ft. per sec.

When will the stone be 800 feet below the point of projection?

$s = ut - \frac{1}{2}gt^2 \therefore -800 = 160t - 16t^2 \therefore t^2 - 10t = 50 \therefore t = 13 \cdot 6$ secs.

38. Let $u =$ velocity of projection. Then the height to which this velocity will carry the body is obtained from equation (3) of Art. 47. (See also solution to Exercise 7.)

$v^2 = u^2 - 2gs \therefore 0 = u^2 - 2gs \therefore s = \dfrac{u^2}{2g}$, the greatest height.

Let $t =$ No. of seconds taken by each body till they meet. Then the spaces described by the bodies in this time are found from equations (2) and (5) of Art. 14; and since the sum of these spaces = the height, we have

$$\tfrac{1}{2}gt^2 + (ut - \tfrac{1}{2}gt^2) = \frac{u^2}{2g} \therefore t = \frac{u}{2g}$$

The space described by the first body in this time $\dfrac{u}{2g}$ is obtained from equation (2), Art. 14, space $= \frac{1}{2}gt^2$ $= \frac{1}{2}g\left(\dfrac{u}{2g}\right)^2 = \dfrac{u^2}{8g} = \frac{1}{4}$th the whole height.

Again the time taken to fall through $\frac{1}{4}$th the height $= \frac{1}{2}$ the time taken to fall through the whole height (since space varies as square of time—See Art. 15). Therefore the balls will meet after a time equal to one-half that taken to fall through the whole height, and at a point one-fourth of the whole height from the top.

39. By the question, $g = 32 \times 2 \cdot 6$. We employ equation (2), Art. 14. $s = \frac{1}{2}gt^2 \therefore 167 = \frac{1}{2} \times (32 \times 2 \cdot 6) \times t^2 \therefore t = \sqrt{\frac{2 \times 167}{32 \times 2 \cdot 6}} = 2$ secs. nearly.

40. See solution to Exercise 34.

(1) $v^2 = u^2 - 2gs \therefore 0 = 48^2 - 2 \times 32 \times s \therefore s = 36$ feet.

(3) Time of flight $= \frac{2u}{g} = \frac{2 \times 48}{32} = 3$ secs.

(2) Range $= ut = 25 \times 3 = 75$ feet.

41. See solution to Exercise 11.

(1) $v^2 = u^2 - 2g \sin i \, s \therefore 0 = 16^2 - 2 \times 32 \times \frac{1}{2} \times s \therefore s = 8$ feet.

(2) $s = ut - \frac{1}{2}g \sin i \, t^2 = 16 \times 5 - \frac{1}{2} \times 32 \times \frac{1}{2} \times 5^2 = -120$ feet.

That is it will be 120 feet *below* the starting point.

42. Let u = the uniform velocity of the balloon, then as it ascends for $4 \cdot 5$ seconds before the stone is dropped, the height reached $= 4 \cdot 5u$ feet. This height is *below* the point of projection of the stone and therefore must be made negative in the equation. (See solution to Exercise 35.)

(1) $s = ut - \frac{1}{2}gt^2 \therefore -4\frac{1}{2}u = 7u - 16 \times 7^2 \therefore 11\frac{1}{2}u = 16 \times 49 \therefore u = 68 \cdot 17$ feet per second.

(2) Height $= ut = 4 \cdot 5 \times 68 \cdot 17 = 306 \cdot 7$ feet.

43. We first find the velocity with which the stone was projected so as just to reach the top. See solution to Exercise 7, and also Art. 47.

$v^2 = u^2 - 2gs \therefore 0 = u^2 - 2 \times 32 \times 100 \therefore u = 80$ feet per sec.

To find when with this velocity it will reach the height of 80 feet we employ equation (5) Art. 14.

$s = ut - \frac{1}{2}gt^2 \therefore 80 = 80t - 16t^2 \therefore t^2 - 5t = -5 \therefore t = 1 \cdot 38$ sec., or $3 \cdot 66$ sec.

The former answer gives the time when the stone *going up* reaches to within 20 feet of the top, and the latter the time *coming down* when the stone is at the same point.

44. By Art. 47, the time of flight $= \frac{2u}{g} = \frac{2 \times 200}{32} = 12\frac{1}{2}$ secs.

45. By Art. 47, the time of reaching greatest height $= \frac{u}{g} \therefore 6 = \frac{u}{32} \therefore u = 32 \times 6 = 192$ feet per sec.

Or:—The question is equivalent to the following:—"With what velocity must a body be projected upwards in order that it may move for 6 seconds before it stops?" We use equation connecting velocity and time. $v = u - gt \therefore 0 = u - 32 \times 6 \therefore v = 192$ feet per sec.

46. See Art. 47.

47. The stone when dropped has an initial upward velocity of 16 feet per second.

(1) To find its height above the point of projection in 1 second we employ equation (5), Art. 14, connecting space and time.

$s = ut - \frac{1}{2}gt^2 = 16 \times 1 - 16 \times 1^2 = 0$. Therefore the stone in 1 second has returned to the point of projection.

(2) To find when the stone will be 400 yards = 1200 feet *below* the point of projection, we use the same equation, making the space negative $\therefore$

$s = ut - \frac{1}{2}gt^2 \therefore -1200 = 16t - 16t^2 \therefore t^2 - t = 75.$

$\therefore t = \frac{1 \pm 17 \cdot 35}{2} = 9 \cdot 17$ seconds after the stone is dropped.

48. See Arts. 17 and 18. And Introductory Example (*k*) to Chap. II.

15 miles per minute gained per minute.

= 15 × 5280 feet per minute gained per minute.

$= \frac{15 \times 5280}{60 \times 60}$ feet per second gained per second.

= 22 feet per second per second.

49. See Arts. 17 and 18.

32 feet per second gained per second.

$= \frac{32}{15}$ units of length per second gained per second.

$= \frac{32 \times 300 \times 30}{15}$ units of length per unit of time per unit of time.

= 192000 units.

CHAPTER V.

EXAMINATION QUESTIONS.

1. If a body possess a given uniform velocity in one direction and a given accelerated velocity in another, show how the path of the body may be determined.

2. What is a *projectile?* What hypothesis is made in the theoretical investigation of the motion of projectiles?

3. What is a *parabola?* Define the following terms referring to a parabola:—*directrix*, *focus*, *axis*, *vertex*, *latus rectum*, *tangent*, *diameter*, *ordinate*, *abscissa*.

4. How is a curve known to be a parabola?

5. Prove that the path of a projectile is a parabola.

6. Show that the velocity of a projectile at any point of its path is equal to what it would acquire if it fell from rest from the directrix to that point.

7. A body is projected with a velocity u at an angle a to the horizon, find—

   (*a*), the latus rectum of the parabola described;

   (*b*), the time in which the projectile reaches its greatest height;

   (*c*), the greatest height;

   (*d*), the whole time of flight;

   (*e*), the time of descent;

   (*f*), the vertical velocity when it returns to the point of projection;

   (*g*), the horizontal range.

8. Show how the equation to the curve described by a parabola may be found.

9. Show how to find the angle at which a body must be projected with a given velocity so as to hit a given point.

10. One body is projected vertically with a velocity u, and another is projected with the same velocity at an angle α to the horizon; compare the relations between space, time, and velocity in the one case with those in the other.

INTRODUCTORY EXAMPLES.

(*a*) A body is projected with a velocity of 500 feet per second at an angle of 30° to the horizon; find the vertical velocity and the horizontal velocity in 5 seconds.

(1) By Art. 59, the vertical velocity is obtained from the equation,

$$v = u \sin \alpha - gt = 500 \sin 30^\circ - 32 \times 5 = 500 \times \tfrac{1}{2} - 32 \times 5$$
$$= 90 \text{ feet per sec.}$$

(2) By Art. 56, the horizontal component $= u \cos \alpha$, and it remains the same throughout the whole flight.

$$\therefore \text{Horizontal velocity} = u \cos 30^\circ = 500 \times \frac{\sqrt{3}}{2} = 250\sqrt{3}$$
feet per sec.

(*b*) When will the projectile mentioned in the last question reach its greatest height? How high will it then be above the horizontal plane passing through the point of projection?

(1) From Art. 59, $v = u \sin \alpha - gt$. And when the body reaches its greatest height the vertical velocity v becomes 0. (See Art. 47.)

$$\therefore 0 = 500 \sin 30^\circ - 32t \therefore 32t = 250 \therefore t = 7\tfrac{13}{16} \text{ secs.}$$

(2) From Art. 60, $v^2 = u^2 \sin^2 \alpha - 2gs$. And when the body reaches its greatest height, v becomes 0. (See Art. 47.)

$$\therefore 0 = 500^2 \sin^2 30^\circ - 2 \times 32 \times s \therefore s = \frac{500 \times 500 \times \frac{1}{4}}{2 \times 32}$$
$$= 976\tfrac{9}{16} \text{ feet.}$$

SOLUTIONS OF EXERCISES TO CHAP. V. PAGE 50.

1. The vertical effects are not affected by the horizontal velocity, therefore by Art. 60.

(*a*) Greatest height $= \dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{32^2 \times \left(\dfrac{\sqrt{3}}{2}\right)^2}{2 \times 32} = 12$ feet.

(*b*) By Art. 61,

$$\text{Time of flight} = \frac{2u \sin \alpha}{g} = \frac{2 \times 32 \times \sin 60^\circ}{32} = \frac{2 \times 32 \times \dfrac{\sqrt{3}}{2}}{32}$$

$= \sqrt{3}$ sec.

(*c*) By Art. 64, the range = horizontal velocity × time of flight. The horizontal velocity = velocity of train + velocity of projection resolved horizontally $= 50 + 32 \cos 60^\circ = 50 + 32 \times \frac{1}{2} = 66$ feet per second.

The time of flight $= \sqrt{3}$ sec. ∴ The horizontal range $= 66 \times \sqrt{3} = 66\sqrt{3}$ feet.

2. To find the vertical height we employ equation (2), Art. 66. Let y = height ∴ $y = u \sin \alpha\, t - \frac{1}{2}gt^2 = 120 \times \sin 30^\circ \times 2 - \frac{1}{2} \times 32 \times 2^2 = 120 - 64 = 56$ feet.

3. We employ equation (1), Art. 66. Let x = the distance

$\therefore x = u \cos \alpha\, t = 120 \times \cos 30^\circ \times 3 = 120 \times \dfrac{\sqrt{3}}{2} \times 3 = 180\sqrt{3}$ feet.

4. See Art. 56. Since the vertical effects are not affected by the horizontal velocity, the time taken to reach the ground is the same in each case, and is the time taken by a body to fall from rest through the height of the tower. Hence by equation (2), Art. 14,

$$(1)\quad s = \tfrac{1}{2}gt^2 \therefore t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2 \times 64}{32}} = 2 \text{ secs.}$$

(2) By Art. 64, the horizontal distance traversed in each case = horizontal velocity × time of flight ∴ the distances are respectively:—$80 \times 2 = 160$ feet; $100 \times 2 = 200$ feet; $120 \times 2 = 240$ feet.

5. Since the direction of motion is horizontal when the projectile is crossing the wall, the top of the wall is at the middle point of the path, and the time of flight = twice the time taken to cross the wall $= 4 \times 2 = 8$ seconds. By Art. 61, the time of flight $= \frac{2u \sin \alpha}{g}$ $\therefore 8 = \frac{2u \sin 45^\circ}{32} = \frac{2 \times u \times \frac{1}{2}\sqrt{2}}{32}$

$\therefore u = \frac{8 \times 32}{2\sqrt{2}} = 128\sqrt{2}$ feet per sec.

6. See solution to Exercise 4. The vertical effects are independent of the horizontal, and the time taken by the cannon ball to reach the ground is the same as that taken to fall from rest through the vertical height of the hill. Hence, by equation (2) Art. 14, we find the height.

$s = \frac{1}{2}gt^2 = \frac{1}{2} \times 32 \times 5^2 = 400$ feet.

7. See solution to Exercise 4.

Horizontal distance = hor. velocity × time of flight $= 1500 \times 5 = 7500$ feet.

8. (1) By Art. 60,

Greatest height $= \frac{u^2 \sin^2 \alpha}{2g} = \frac{256^2 \times (\frac{1}{8})^2}{2 \times 32} = 16$ feet.

(2) By Art. 61,

Time of flight $= \frac{2u \sin \alpha}{g} = \frac{2 \times 256 \times \frac{1}{8}}{32} = 2$ secs.

9. (1) By Art. 61,

Time of flight $= \frac{2u \sin \alpha}{g} = \frac{2 \times 256 \times \frac{1}{2}}{32} = 8$ secs.

(2) The question is now resolved into the following:—With what vertical velocity must a body be projected in order that its time of flight may be 8 secs.? The Time of vertical flight is, by Art. 47, $= \frac{2u}{g} \therefore \frac{2u}{g} = 8$

$\therefore u = \frac{8 \times 32}{2} = 128$ feet per second.

CHAPTER VI.

EXAMINATION QUESTIONS.

1. State the relations which subsist between the mass of a body, the force acting upon it, and the acceleration produced.

2. Give Newton's Second Law of Motion.

3. Explain fully how the relations mentioned in question (1) may be experimentally verified.

4. How may Atwood's Machine be employed to verify the laws of falling bodies?

5. Show how the inclined plane may be used for the same purpose.

6. Give the algebraic formula which expresses the relations between force, mass, and acceleration; and show how it is obtained.

7. Show from this formula that in a vacuum all bodies light and heavy fall with the same velocity.

8. Supposing it proved by experiment that all bodies fall in a vacuum with the same velocity, show that the weights of bodies are proportional to their quantities of matter.

9. What are the standard force and corresponding acceleration with which other forces and their accelerations may be compared?

10. What are gravitation units of force? Show how the corresponding units of mass are determined.

11. State the advantages and disadvantages in the employment of gravitation units of force.

12. What are absolute or kinetic units of force, and how are they obtained?

13. What are the C.G.S. units?

14. What is the British, and what the French standard of mass?

15. Give the names of the chief gravitation units of force, and the chief absolute units of force in use.

16. How may quantities expressed in gravitation units be reduced to their equivalents in kinetic units, and conversely?

17. Define momentum. How is momentum measured?

18. What is the kinetic measure of a force, and how may it be expressed?

19. How is the effect of a force acting for a given time measured?

20. When a body moves against a resisting force, state the relation which subsists between the momentum, the resistance, and the time for which the body moves.

21. Making use of the standard force and its corresponding acceleration, determine the acceleration in each of the following cases:—

(1) When a force acts on a body which is perfectly free to move.

(2) When two unequal bodies are connected by a string which passes over a pulley turning in a vertical plane.

(3) When one body hanging vertically draws another along a smooth horizontal table.

(4) When two inclined planes have a common vertex, and a body on one of the planes draws another body up the other plane by means of a string passing over the common vertex of the planes.

22. How may the relations between space, time, and acceleration be determined in the foregoing cases?

INTRODUCTORY EXAMPLES TO CHAP. VI.

(*a*) A force of 64 lbs. acts on a mass of 128 lbs.: What is the acceleration?

By Art 75, $f=\frac{F}{m}$. Then if we employ gravitation units, and take the unit of force = 1 lb. weight, the given force of 64 lbs. = 64 units of force (see Art. 80), and the given mass 160 lbs. = $\frac{160}{32}$ units of mass.

Substituting in equation of Art. 75, we have $f=\frac{F}{m}=\frac{64}{\frac{128}{32}}=16$ feet per sec. per sec.

Or:—If we employ absolute units, and take the mass of 1 lb. as the unit of mass, then (Art. 82) the unit of force is $\frac{1}{32}$ lb. weight, and therefore the given force 64 lbs. $= 64 \times 32$ units of force, and the given mass 128 lbs. = 128 units of mass. ∴ Art. 75, $f=\frac{F}{m}=\frac{64 \times 32}{128}=16$ feet per sec. per sec.

Or:—The problem may also be solved by the method of Art. 90 (1), where the reductions of the two former methods are not required. See Art. 78. If a force of 128 lbs. acted on the given mass of 128 lbs. the acceleration would be $g=32$, what will be the acceleration when a force of 64 lbs. acts on the same mass?

As $128:64::32:f \therefore f=16$ feet per sec. per sec.

(*b*) A force of 8 lbs. produces an acceleration of 4 feet per second per second in a certain mass: find the mass.

By Art. 75, $F=mf \therefore m=\frac{F}{f}$. Using gravitation units, the given force of 8 lbs. = 8 units of force, and the given acceleration = 4 units of acceleration ∴ m (units of mass) $=\frac{8}{4}=2$. Therefore the mass required = 2 gravitation units of mass. But, Art. 80, each of these units = 32 lbs. ∴ mass $= 2 \times 32 = 64$ lbs.

Or:—Using absolute units,

m (units of mass) $=\frac{8 \times 32}{4}=64$ ∴ The required mass = 64 absolute units of mass. But, Art. 82, each of these units = 1 lb., therefore, the mass = 64 lbs.

Or :—Employing the principle of Art. 70 (3).—The force 8 lbs. acting on the mass of 8 lbs. produces an acceleration 32 (See Art. 78), it is required to determine the mass upon which, if the same force act, it will produce an acceleration 4.

As, Acceleration 4 : acceleration 32 : : mass 8 lbs. : ? mass.

The fourth term of the proportion is found to be 64 lbs., which is therefore the required mass.

(*c*) A force of 12 lbs. acts on a mass of 48 lbs. for 3 seconds : find the momentum produced.

The acceleration may be found by the equation of Art. 75, or by the method of Art. 90 (1). See Example (*a*).

Using the latter method we have

As 48 lbs. force : 12 lbs. force : : 32 : f $\therefore f = 8$ feet per 1″ per 1″.

The velocity in 3 seconds is found from equation (1), Art. 12, $v = ft$ $\therefore v = 8 \times 3 = 24$ ft. per 1″. And by Art. 86, momentum = mass × velocity. Now 48 lbs. of mass $= \frac{48}{32}$ gravitation units of mass $\therefore$ momentum

$$= mv = \frac{48}{32} \times 24 = 36.$$

(*d*) If a body weighing 8 lbs. possess a velocity of 40 feet per second, how long will it move against a resisting force of 10 lbs. ?

Using gravitation units, the body of 8 lbs. has a mass of $\frac{8}{32}$ units of mass. By Art. 89, $t = \frac{mv}{R}$, therefore

$$t = \frac{\frac{8}{32} \times 40}{10} = 1 \text{ second.}$$

Or :—Using absolute units, the given body has a mass of 8 units, and the resistance = 10 × 32 units of force

$$\therefore \text{Art. } 89,\ t = \frac{mv}{R} = \frac{8 \times 40}{10 \times 32} = 1 \text{ second.}$$

SOLUTIONS OF EXERCISES TO CHAP. VI. PAGE 64.

1. The acceleration is given by equation of Art. 90 (1),

$$f = \frac{P}{Q}g = \frac{10}{40} \times 32 = 8 \text{ feet per } 1'' \text{ per } 1''.$$

This solution depends on the principles stated in Art. 70 (2) and Art. 78. A force of 40 lbs. acting on the mass of 40 lbs. would produce an acceleration 32, what acceleration will a force of 10 lbs. produce on the same mass?

As, 40 lbs. force : 10 lbs. force : : acceleration 32 : ? acceleration.

The required acceleration is found as above to be 8 feet per 1″ per 1″.

We may also find the acceleration by means of the equation of Art. 75; but in using it we must, if we employ gravitation units, reduce the given *mass* to its equivalent expressed in gravitation units of mass; or, if we use absolute units, we must reduce the given *force* to its equivalent in absolute units of force. Hence (1) using gravitation units, the given mass of 40 lbs. $= \frac{40}{32}$ gravitation units of mass (See Art. 80); and since 1 lb. = the gravitation unit of force, the given force 10 lbs. = 10 units of force,

$$\therefore \text{(Art. 75)}, \ f = \frac{F}{m} = \frac{10}{\frac{40}{32}} = 8 \text{ feet per } 1'' \text{ per } 1''.$$

Or (2) :—Using absolute units, the given force of 10 lbs. = 10 × 32 absolute units of force (See Art. 82); and since the absolute unit of mass = 1 lb. mass, the given mass of 40 lbs. = 40 units of mass ∴ Art. 75,

$$f = \frac{F}{m} = \frac{10 \times 32}{40} = 8 \text{ feet per } 1'' \text{ per } 1''.$$

2. (1) By Art. 90 (2), $f=\frac{P-Q}{P+Q}g=\frac{17-15}{17+15}\,32=2$ feet per 1″ per 1″.

(2) To find space described with this acceleration we employ equation (2), Art. 12,

$s=\frac{1}{2}ft^2=\frac{1}{2}\times 2\times 5^2=25$ feet.

The acceleration may also be determined, as in the preceding solution, by means of the equation of Art. 75. But in using this equation, if we employ gravitation units, the given quantities of force and mass must be reduced to their equivalents, expressed in gravitation units; and if we employ absolute units, the given quantities must be expressed in absolute units.

(1) Using gravitation units, and taking 1 lb. weight as the unit of force, then, Art. 80, the unit of mass $=32$ lbs. mass. Therefore the force which produces motion in the system $=\left(\frac{17}{16}-\frac{15}{16}\right)=\frac{1}{8}$ gravitation unit of force. And the mass moved $=\frac{\frac{17}{16}+\frac{15}{16}}{32}=\frac{1}{16}$ gravitation unit of mass.

$$\therefore \text{Art. } 75, f=\frac{F}{m}=\frac{\frac{1}{8}}{\frac{1}{16}}=2 \text{ feet per 1″ per 1″}.$$

Or (2), using absolute units, and taking 1 lb. mass as the unit of mass, then (Art. 82), the unit of force $=\frac{1}{32}$ lb. weight. Therefore the force which produces motion in the system $= \left(\frac{17}{16} - \frac{15}{16}\right) \times 32 = \frac{1}{8} \times 32 = 4$ absolute units of force. And the mass moved $= \left(\frac{17}{16} + \frac{15}{16}\right) = 2$ absolute units of mass.

$\therefore$ Art. 75, $f = \frac{F}{m} = \frac{4}{2} = 2$ feet per 1″ per 1″.

NOTE.—In the above the weight of 1 oz. might have been taken as the gravitation unit of force, and the mass of 1 oz. as the absolute unit of mass, and the division by 16 would not then be required.

3. We employ equation (3), Art. 12,

$$v^2 = 2fs = 2 \times 2 \times 10 \therefore v = \sqrt{40} \text{ feet per sec.}$$

4. We employ equation (1), Art. 12,

$$v = ft = 2 \times 3 = 6 \text{ feet per sec.}$$

5. (1) By Art. 90 (3), $f = \frac{P}{P+Q}g = \frac{10}{10+30} \times 32 = 8$ feet per 1″ per 1.″

(2) And by equation (2), Art. 12,

$$s = \tfrac{1}{2}ft^2 = \tfrac{1}{2} \times 8 \times 2^2 = 16 \text{ feet.}$$

The acceleration may also be found by equation of Art. 75, as in solutions to Exercises 1 and 2.

(1) Using gravitation units, and taking 1 lb. weight as the gravitation unit of force, the force which produces motion in the system is 10 lbs. = 10 gravitation units of force. And since the gravitation unit of mass = the mass of 32 lbs. $\therefore$ the mass moved $= (10+30)$ lbs. $= \frac{10+30}{32}$ gravitation units of mass.

Hence, Art. 75, $f = \frac{F}{m} = \dfrac{10}{\frac{10+30}{32}} = 8$ feet per 1″ per 1″.

(2) Using absolute units, and taking the mass of 1 lb. = the unit of mass ∴ the mass moved = (10 + 30) lbs. = (10 + 30) absolute units of mass. And the force which produces motion in the system = 10 lbs. = 10 × 32 absolute units of force (See Art. 82).

Hence, Art 75, $f = \frac{F}{m} = \frac{10 \times 32}{10 + 30} = 8$ feet per 1″ per 1″.

6. We first find the velocity at the end of 2 seconds, by equation (1), Art 12. $v = ft = 8 \times 2 = 16$ feet per sec.

(1) The body on the smooth table moves *uniformly* with the velocity it has acquired ∴ Art. 6, space described $= ut = 16 \times 3 = 48$ feet.

(2) The body moving downwards has an initial velocity of 16 feet per second when the string breaks, and it then receives an acceleration of 32 ft. per 1″ per 1″ from gravity ∴ we use equation (5), Art. 14.

$s = ut + \frac{1}{2}gt^2 = 16 \times 3 + \frac{1}{2} \times 32 \times 3^2 = 192$ feet.

7. We first find the acceleration. And since the body with a certain acceleration moves from rest through a space of $2\frac{1}{2}$ feet in 1 second ∴ equation (2), Art. 12, $s = \frac{1}{2}ft^2$ $\therefore 2\frac{1}{2} = \frac{1}{2}f \times 1^2 \therefore f = 5$ feet per 1″ per 1″.

Again, Art. 90 (1). Since forces are proportional to their accelerations when the mass is constant, and since 8 lbs. force, acting on the mass of 8 lbs., will produce an acceleration 32, we have to find what force will produce an acceleration 5 in the same mass. Let P lbs. be the force. Then, as 8 lbs. force : P lbs. : : 32 : 5 ∴ $P = 1\frac{1}{4}$ lbs.

The force may also be obtained from the equation of Art. 75. $F = mf$. And using gravitation units, $F = \frac{8}{32} \times 5 = 1\frac{1}{4}$ lbs. Or, using absolute units,

$F = 8 \times 5 = 40$ absolute units of force $= \frac{40}{32}$ lbs. $= 1\frac{1}{4}$ lbs.

8. Since 16 is the measure of the force—that is, the number which expresses the force—of 16 lbs. weight, therefore the unit of force = 1 lb. weight. Hence, by Art. 80, the unit of mass = the mass of 32 lbs. Therefore, 16 lbs. mass contains $\frac{16}{32} = \frac{1}{2}$ unit of mass. Therefore, the number which expresses the mass of 16 lbs., when the mass of 32 lbs. is the unit of mass, is $\frac{1}{2}$.

9. Using the equation of Art. 90 (1), $f = \frac{P}{Q}g$.

$\therefore$ (1) $f = \frac{5}{5} \times 32 = 32$ feet per 1″ per 1″

(2) $f = \frac{5}{20} \times 32 = 8$ „ „ „

(3) $f = \frac{20}{5} \times 32 = 128$ „ „ „

These results may also be obtained from equation of Art. 75, as in solutions to Exercises (1) and (2).

10. We use equation of Art. 90 (1), to find the acceleration.

$$f = \frac{P}{Q}g = \frac{7}{112} \times 32 = 2 \text{ feet per } 1'' \text{ per } 1''$$

(1) To find velocity :—$v = ft = 2 \times 5 = 10$ feet per sec.

(2) To find space :—$s = \frac{1}{2}ft^2 = \frac{1}{2} \times 2 \times 10^2 = 100$ feet.

11. See solution to Exercise 7.

To find the acceleration :—$s = \frac{1}{2}ft^2 \therefore 5 = \frac{1}{2} \times f \times 1^2$ $\therefore f = 10$ ft. per 1″ per 1″.

Let P = the force, and employing the method of Art. 90 (1),

8 lbs. force : P lbs. force : : 32 accltn. : 10 accltn.

$\therefore P = 2\frac{1}{2}$ lbs.

The force may also be found by the equation of Art. 75.

12. When the force ceases to act on the body it moves uniformly with the velocity it had acquired by the action of the force. Therefore since it moves over 60 feet in 2 seconds with a uniform velocity, that velocity must be 30 feet per second. Hence—

(1) To find the acceleration, we employ equation (1), Art. 12.

$$v = ft \therefore f = \frac{v}{t} = \frac{30}{5} = 6 \text{ feet per } 1'' \text{ per } 1''.$$

(2) To find the force in gravitation units, we may use either the method of Art. 90 (1), or the equation of Art. 75 (see solution to Exercise 1). Using the latter method and taking 1 lb. as the unit of force then the 50 lbs. of mass $= \frac{50}{32}$ units of mass by Art. 80. Hence

$$F = mf \therefore F \text{ (in lbs. weight)} = \frac{50}{32} \times 6 = 9\tfrac{3}{8} \text{ lbs.}$$

(3) Using absolute units, and taking 1 lb. mass as the unit of mass, then the given mass of 50 lbs. = 50 units of mass. Hence by Art. 75, $F = mf \therefore F$ (in absolute units) $= 50 \times 6 = 300$ absolute units of force.

13. We employ equation (4) of Art. 14, connecting velocity and time, and using the - sign; $v = u - gt$. And since the velocity is *downwards* at the required time, we have

$$-40 = 96 - 32t \therefore 32t = 136 \therefore t = 4\tfrac{1}{4} \text{ seconds.}$$

14. We find the acceleration by Art. 90 (3).

$$f = \frac{P}{P+Q}g = \frac{20}{20+100} \times 32 = \frac{16}{3} \text{ feet per } 1'' \text{ per } 1''.$$

To find space we use equation (3), Art. 12,

$$s = \tfrac{1}{2}ft^2 = \tfrac{1}{2} \times \frac{16}{3} \times 5^2 = 66\tfrac{2}{3} \text{ feet.}$$

NOTE.—The acceleration may also be found by equation of Art. 75, as in solution to Exercise (1).

15. We first find the acceleration with which 64 ft. is described in 5 secs.

$$s=\tfrac{1}{2}ft^2 \therefore 64=\tfrac{1}{2}f\times 5^2 \therefore f=\frac{128}{25}\text{ feet per 1'' per 1''.}$$

We now employ equation of Art. 90 (2), $f=\dfrac{P-Q}{P+Q}g$.

Let $P = 50$ ozs. + required additional weight, and $Q = 50$ ozs. Then $\dfrac{128}{25}=\dfrac{P-50}{P+50}\times 32$; and solving for P, we obtain $P=69\frac{1}{21}$ ozs., from which deducting 50 ozs. we have $19\frac{1}{21}$ ozs., the required weight.

16. By Art. 90 (2); and letting $P = 10$ lbs. and $Q = x$ lbs., the required weight,

$$f=\frac{P-Q}{P+Q}g \therefore 16=\frac{10-x}{10+x}\times 32 \therefore x=3\tfrac{1}{3}\text{ lbs.}$$

Another answer would be obtained by putting $x =$ the greater weight.

17. We find the acceleration with which the system moves, by the equation of Art. 90 (2).

$$f=\frac{P-Q}{P+Q}g=\frac{10-8}{10+8}\times 32=3\tfrac{5}{9}\text{ feet per 1'' per 1''.}$$

18. If there were no friction, the force P required to give an acceleration of 5 feet per 1″ per 1″ to a body of 18 lbs. would be obtained from the principles of Art. 70 (2), and Art. 78. A force of 18 lbs. acting on a mass of 18 gives an acceleration 32, what force acting on the same mass will give an acceleration of 5? Let P = force, then

As 18 lbs. : P lbs. : : 32 : 5 $\therefore P=2\frac{13}{16}$ lbs.

To this must be added the force necessary to overcome friction,

Therefore $2\frac{13}{16}+2=4\frac{13}{16}$ lbs., the required force.

19. (1) To find acceleration with which the system describes 20 feet in $2\frac{1}{2}$ seconds.

$s = \frac{1}{2}ft^2 \therefore 20 = \frac{1}{2}f \times (2\frac{1}{2})^2 \therefore f = \frac{32}{5}$ feet per 1″ per 1″.

(2) By Art. 90 (2),

$f = \frac{P - Q}{P + Q} \cdot g$. Let x = required weight, then by question,

$$\frac{32}{5} = \frac{50 + x - (50 - x)}{50 + x + 50 - x} \times 32 \therefore \frac{32}{5} = \frac{2x}{100} \times 32 \therefore x = 10 \text{ lbs.}$$

20. (1) We find the acceleration by Art. 90 (3)

$f = \frac{P}{P + Q} g = \frac{8}{8 + 12} \times 32 = \frac{64}{5}$ feet per 1″ per 1″.

(2) The space in 3rd second = space in 3 seconds − space in 2 seconds $= \frac{1}{2}f(3^2 - 2^2) = \frac{1}{2} \times \frac{64}{5} \times 5 = 32$ feet. (See Exam. 2, Chap. II.)

21. (1) To find the acceleration we employ equation of Art. 90 (4).

$$f = \frac{12 \sin 60^\circ - 4 \sin 30^\circ}{12 + 4} g = \frac{12 \times \frac{\sqrt{3}}{2} - 4 \times \frac{1}{2}}{12 + 4} \times 32 =$$

$12\sqrt{3} - 4$ feet per 1″ per 1″.

(2) To find the velocity we employ equation (1), Art. 12, $v = ft = (12\sqrt{3} - 4)\,4 = 67 \cdot 136$ feet per sec.

22. We use equation (2), Art. 12,

$s = \frac{1}{2}ft^2 = \frac{1}{2}(12\sqrt{3} - 4) \times 4^2 = 134 \cdot 272$ feet.

23. (1) The acceleration is obtained from equation (2) Art. 12,

$s = \frac{1}{2}ft^2$. It will be more convenient in this case to take 1 mile as the unit of length, and 1 hour as the unit of time $\therefore$ 1 mile $= \frac{1}{2}f \times \left(\frac{3}{60}\right)^2 \therefore f = 800$ miles per hour per hour.

(2) To find the velocity we employ equation (1), Art. 12,

$$v = ft = 800 \times \frac{3}{60} = 40 \text{ miles per hour.}$$

(3) To find space, we use equation (2), Art. 12,

$$s = \tfrac{1}{2}ft^2 = \tfrac{1}{2} \times 800 \times \left(\frac{2}{60}\right)^2 = \frac{4}{9} \text{ mile.}$$

NOTE.—The same result will be obtained by taking 1 foot as the unit of length, and 1 second the unit of time. With these units $f = \frac{44}{135}$ ft. per 1″ per 1″.

24. Forces are proportional to their accelerations, when acting on the same mass, Art. 70 (2). Therefore since the weight of the train would produce in the mass of the train an acceleration 32 (Art. 78) the force that would produce the acceleration $\frac{44}{135}$ is given by the proportion of Art. 78.

Force of engine : weight of train :: $\frac{44}{135}$: 32

:: 11 : 1080.

25. (1) If in Art. 90 (4), P hang vertically then the force which produces motion in the system is $P - Q \sin i'$ ∴ as in Art. 90 (4),

$$f = \frac{P - Q \sin i'}{P + Q} g = \frac{5 - 6 \times \frac{2}{9}}{5 + 6} \times 32 = \frac{32}{3} \text{ ft. per 1″ per 1″.}$$

(2) To find the velocity after moving through the space of 4 feet, $v^2 = 2fs = 2 \times \frac{32}{3} \times 4 \therefore v = 9 \cdot 23$ feet per sec.

26. (1) The force which causes motion along the plane = 300 − weight of waggon resolved along the plane $= 300 - 1120 \sin i = 300 - 1120 \times \frac{20}{200} = 188$ lbs. The weight of the waggon would produce in the mass of the waggon an acceleration 32, therefore 188 lbs. will produce an acceleration f, given by the following proportion (Art. 78).

$1120 : 188 :: 32 : f \therefore f = \frac{188}{35}$ feet per 1″ per 1″.

(2) To find the time,

$s = \frac{1}{2} ft^2 \therefore 200 = \frac{1}{2} \times \frac{188}{35} \times t^2 \therefore t = 8 \cdot 6$ secs.

27. The acceleration produced by the force is obtained from Art. 90 (1), $f = \frac{P}{Q} g = \frac{1}{16} \times 32 = 2$ feet per 1″ per 1″.

(2) Now since the body has a velocity of 20 ft. per 1″ and it is required to find the space through which it must move with the acceleration 2 feet per 1″ per 1″ that its velocity may become 30 feet per second, we employ equation (3), Art. 11, with the + sign $\therefore$

$v^2 = u^2 + 2fs \therefore 30^2 = 20^2 + 2 \times 2 \times s \therefore s = 125$ feet.

NOTE.—The answer in the Text-book, p. 220, No. 27 of Exs. to Chap. VI., should be 125 feet.

28. From Art. 89, Resistance $= \frac{\text{momentum}}{\text{time}} = \frac{mv}{t}$

$$= \frac{\frac{12}{32} \times 64}{18} = 1\frac{1}{3} \text{ lbs.}$$

NOTE.—Since the resistance is found in gravitation units the given mass of 12 lbs. $= \frac{12}{32}$ units of mass. (See Art. 80.)

29. (1) To find the acceleration, we may employ the equation of Art. 90 (2), or that of Art. 75. Taking the former, we have

$f = \frac{P - Q}{P + Q} g = \frac{5 - 4}{5 + 4} \times 32 = \frac{32}{9}$ feet per 1″ per 1″.

(2) To find the velocity after 3″ with this acceleration,

$$v = ft = \frac{32}{9} \times 3 = \frac{32}{3} \text{ feet per sec.}$$

(3) When the bar is removed the weight moving upwards is 1 lb. heavier than the other, hence a force of 1 lb. acts as a resistance tending to bring the system to rest. Therefore Art. 89,

$$t = \frac{mv}{R} = \frac{\frac{3+4}{32} \times \frac{32}{3}}{1} = \frac{7}{3} = 2\tfrac{1}{3} \text{ secs.}$$

Note.—As in the last solution the moving mass, = (3 + 4) lbs., is reduced to gravitation units of mass by dividing by 32.

30. The system comes to rest for an instant, and it then begins to move in the opposite direction, the 4 lbs. weight descending and the 3 lbs. ascending. The acceleration is determined as in the last case by the equation of Art. 90 (2).

(1) $f = \frac{P - Q}{P + Q} g = \frac{4 - 3}{4 + 3} \times 32 = \frac{32}{7}$ feet per 1″ per 1″.

(2) The velocity after 7 seconds is obtained from equation (1), Art. 12,

$$v = ft = \frac{32}{7} \times 7 = 32 \text{ feet per second.}$$

31. (1) The acceleration (see solution to Ex. 29)

$$= \frac{32}{9} \text{ feet per } 1'' \text{ per } 1''.$$

(2) To find velocity after 9 seconds,

$$v = ft = \frac{32}{9} \times 9 = 32 \text{ feet per second.}$$

(3) By Art. 89,

$$t = \frac{mv}{R} = \frac{\frac{3+4}{32} \times 32}{1} = 7 \text{ secs.}$$

32. See Art. 86.

33. See Art. 77. The earth attracts all kinds of matter alike, and hence the weights of bodies are proportional to their masses. Therefore if m be any mass and W its weight, m' any other mass and W′ its weight, then $m : m' :: W : W'$. Now if (Art. 76) the weight W acting on the mass m produce an acceleration g, the same acceleration will be produced by the weight W′ acting on the mass m', for, Art. 76, $g = \frac{W}{m} = \frac{W'}{m'}$. The acceleration being directly proportional to the force and inversely proportional to the mass must be the same in both cases.

34. The unit of force is in this case the force which will produce in 1 lb. mass an acceleration of 1 foot per 1″ per 1″. By Art. 82 this force $= \frac{1}{g} = \frac{1}{32}$ lb. In $1\frac{1}{2}$ lbs. there are $\frac{1\frac{1}{2}}{\frac{1}{32}} = 48$ of these units.

35. A force of 100 lbs. will produce in—
mass of 100 lbs. an acceleration of 32 feet per 1″ per 1″; and in
mass of 100×16 lbs. „ 2 „ „ „ and in
„ $\frac{100 \times 16}{4 \times 4}$ lbs. „ 2 feet per $\frac{1}{4}$″ per $\frac{1}{4}$″.

Therefore required mass = 100 lbs.

Or : 2 feet per $\frac{1}{4}$″ per $\frac{1}{4}$″

$= 2 \times 4 \times 4 = 32$ feet per 1″ per 1″ (Arts. 17 and 18).

But this acceleration will be produced by the given force, 100 lbs. in the mass of 100 lbs. (Art. 78). Therefore the unit of mass = 100 lbs. if the units of force, length, and time be those given in the question.

36. The unit force being (Art. 79) that force which will produce in the unit mass the unit acceleration, it follows

that the unit force required is the force which will produce
in mass 2240 lbs. an accltn. of 1 yd. per min. per min.;
or in mass 2240 × 3 lbs. " 1 foot " "
or in " $\frac{2240 \times 3}{32}$ lbs. " 32 feet " "
or in " $\frac{2240 \times 3}{32 \times 60 \times 60}$ lbs. " 32 feet per sec. per sec.

This mass $= \frac{7}{120}$ lb. Therefore the unit force must produce in the mass $\frac{7}{120}$ lb. an acceleration of 32 feet per 1″ per 1″, and (Art. 78) the force $= \frac{7}{120}$ lb. weight. This force, therefore, will produce in the mass of one ton an acceleration of 1 yard per minute per minute.

CHAPTER VII.

EXAMINATION QUESTIONS.

1. Define the terms *action*, *reaction*, and *stress*.

2. Define *pressure*, *tension*, *attraction*.

3. State Newton's Third Law of Motion, and show how it is obtained.

4. Give Newton's illustrations of the Third Law.

5. Determine the tension of the string in Atwood's Machine.

6. Determine the tension of the string: (*a*), when a weight hanging vertically draws another weight along a smooth table; (*b*), when a weight moving down an inclined plane draws a weight up another inclined plane.

7. When a weight rests upon a plane determine the pressure upon it: (*a*), when the plane is at rest or moving uniformly; (*b*), when the plane is moving upwards with a given acceleration; (*c*), when it is moving downwards with a given acceleration.

GENERAL REMARKS ON THE LAWS OF MOTION.

1. State the Three Laws of Motion.

2. How is the truth of these laws established?

3. Point out the connexion of these laws with force.

4. Give an explanation of the first law. Show that the law supplies a definition of *force* and of *equal times*.

5. Give an explanation of the second law. What are *equal forces?* What are *equal masses?*

6. What inferences may be drawn from the *silence* of the law on certain points?

7. Show that forces may be compounded in the same way as accelerations.

8. Give an explanation of the third law. What important extension may be given to this law?

INTRODUCTORY EXAMPLES TO CHAPTER VII.

(*a*) Two bodies each weighing 4 ozs. are connected by a string passing over a pulley, as in Atwood's Machine: find the tension of the string.

By Art. 97, $T = \frac{2PQ}{P+Q} = \frac{2 \times 4 \times 4}{4+4} = 4$ ozs.

Or:—Each body is acted on by two forces, its weight and the tension of the string; and since the bodies are at rest, the forces acting on each weight must be equal and opposite. Hence the tension = 4 ozs.

(*b*) In the arrangement mentioned in the preceding question a bar weighing 2 ozs. is placed on one of the bodies: find the tension of the string while the system is in motion.

By Art. 97, $T = \frac{2PQ}{P+Q} = \frac{2 \times 6 \times 4}{6+4} = 4 \cdot 8$ ozs.

(*c*) A weight of 1 lb. hanging vertically draws a weight of 3 lbs. along a smooth table: find the tension on the string.

By Art. 98, $T = \frac{PQ}{P+Q} = \frac{1 \times 3}{1+3} = \frac{3}{4}$ lb.

(*d*) A man whose weight is 140 lbs. stands on a lift: what is the pressure on the lift (1) when it is ascending uniformly with a velocity of 8 feet per 1″? (2) when it is descending with an acceleration of 8 feet per 1″ per 1″?

(1) By Art. 100, when the lift is ascending or descending *uniformly* it prevents an acceleration in the weight of 32 feet per 1″ per 1″, and therefore supports a pressure which would produce this acceleration. That pressure = weight of body = 140 lbs.

(2) The lift now prevents an acceleration of only 24 feet per 1″ per 1″, and therefore supports a pressure which would produce in 140 lbs. this acceleration.

Calling P the pressure, and since 140 lbs. weight would produce in the mass of 140 lbs. an acceleration 32, the weight that would produce an acceleration 24 is obtained from the proportion of Art. 78,

$$32 : 24 :: 140 \text{ lbs.} : \text{P lbs.} \therefore \text{P} = 105 \text{ lbs.}$$

Solutions of Exercises to Chapter VII. Page 74.

1. Let x = the weight. Then by Art. 97,

$$\text{T} = \frac{2\text{PQ}}{\text{P}+\text{Q}} \therefore 12 = \frac{2 \times 10 \times x}{10 + x} = \frac{20x}{10+x} \therefore 120 + 12x = 20x$$
$$\therefore x = 15 \text{ lbs.}$$

2. By Art. 98, $\text{T} = \frac{\text{PQ}}{\text{P}+\text{Q}} = \frac{25 \times 100}{25 + 100} = 20$ lbs.

3. By Art. 98, $\text{T} = \frac{\text{PQ}}{\text{P}+\text{Q}} \therefore 5 = \frac{\text{P} \times 50}{\text{P} + 50} \therefore \text{P} = 5\frac{5}{9}$ lbs.

4. By the Third Law of Motion the momenta are equal and opposite, that is the mass × velocity of the ball = mass × velocity of the cannon. Therefore using absolute units of mass,

$$100 \times 1000 = 2240 \times v \therefore v = 44 \cdot 6 \text{ feet per second.}$$

5. The velocity of one half being increased 60 feet per second, the velocity of the other half must (by Third Law of Motion) be diminished by 60 feet per second. Therefore the velocity of the latter will be $100 - 60 = 40$ feet per second.

6. See Art. 100, and Introductory Example (*d*), Chap. VII.

   (1) When moving uniformly the tension of the rope prevents an acceleration on the cage of 32 feet per 1″ per 1″, and therefore (Art. 78) the tension = the weight of the cage = $\frac{1}{2}$ ton.

(2) In this case the tension of the rope prevents an acceleration downwards of 32, and communicates an acceleration upwards of 10 feet per 1″ per 1″. Hence the whole tension corresponds to an acceleration of $32 + 10 = 42$ feet per 1″ per 1″ $\therefore$ As $32 : 42 :: \frac{1}{2}$ ton : Tension $\therefore$ Tension $= \frac{21}{32}$ ton.

7. (1) Working from first principles as in Art 97, and using gravitation units, we have the acceleration in the case of the 50 grammes $= \dfrac{T - 50}{\frac{50}{g}}$, and for the 60 grammes $= \dfrac{60 - T}{\frac{60}{g}}$. (See Art. 80. The given masses are expressed in gravitation units of mass by dividing by g). Then as in Art. 97, $\dfrac{60 - T}{\frac{60}{g}} = \dfrac{T - 50}{\frac{50}{g}} \therefore T = 54\frac{6}{11}$ grammes.

(2) Using absolute units (see Art. 82) the acceleration in the case of the 50 grammes $= \dfrac{T \text{ (in dynes)} - 50g}{50}$. And the acceleration in the case of the 60 grammes $= \dfrac{60g - T \text{ (in dynes)}}{60} \therefore \dfrac{60g - T}{60} = \dfrac{T - 50g}{50} \therefore T = 54\frac{6}{11}g$ dynes, and since $g = 981 \therefore T = 53{,}509$ dynes.

NOTE.—It will be seen that the result in gravitation units may be reduced at once to that in absolute units by multiplying by g. See Art. 85.

8. The force necessary to *sustain* the weight upon the inclined plane = the effect of the weight resolved along the plane = weight × sine of the plane's inclination = 8 tons $\times \frac{1}{5} = \frac{8}{5}$ ton.

Hence if the weight were at rest or in uniform motion, the tension of the string would be $\frac{8}{5}$ ton. (See Art. 100, and Introductory Examples (*d*), Chapter VII.)

The additional force required to communicate to the mass of 8 tons an acceleration of 3 feet per 1″ per 1″ is obtained from the proportion :—As 32 acceleration : 3 acceleration :: 8 tons : P $\therefore$ P $= \frac{3}{4}$ ton.

And $\frac{8}{5} + \frac{3}{4} = 2\frac{7}{20}$ tons, the whole tension.

9. See Solution to Exercise 29, Chap. VI.

(1) To find the acceleration we employ equation of Art. 90 (2),

$$f = \frac{P - Q}{P + Q} g = \frac{502 - 500}{502 + 500} g = \frac{2g}{1002} \text{ centimetres per } 1'' \text{ per } 1''.$$

(2) To find velocity after 3 seconds.

$$v = ft = \frac{2g}{1002} \times 3 = \frac{g}{167} \text{ centimetres per second.}$$

(3) When the 3 grammes are removed a force of 1 gramme resists the motion. Hence Art. 89,

$$t = \frac{mv}{R} = \frac{\frac{500 + 499}{g} \times \frac{g}{167}}{1} = 5\tfrac{164}{167} \text{ seconds.}$$

10. See Art. 100 and Introductory Example (*d*), Chap. VII.

11. See Art. 100, and Introductory Example (*d*).

(*a*) and (*b*) The pressure is equal to the weight = 140 lbs.

(*c*) As $g : 1\frac{1}{10}g$ :: 140 : 154 lbs.

(*d*) As $g : \frac{9}{10}g$ :: 140 : 126 lbs.

(*e*) The lift prevents no acceleration and therefore bears no pressure.

(*f*) As $g : 1\frac{1}{4}g :: 140 : 175$ lbs.

(*g*) As $g : 2g :: 140 : 280$ lbs.

12. (*a*) As $32 : 36 :: 100 : 112\frac{1}{2}$ lbs.

(*b*) As $32 : 28 :: 100 : 87\frac{1}{2}$ lbs.

13. See Art. 100. The hand does not prevent any acceleration in the weight after the instant of jumping, since both arm and weight descend with the same acceleration, and therefore the hand does not sustain any pressure.

14. See Solution to Exercise 8.

(1) When the velocity is uniform the tension = the component of the weight resolved along the plane

$$= 10 \sin 30^\circ = 10 \times \frac{1}{2} = 5 \text{ tons.}$$

(2) The force acting along the plane required to give an acceleration of 4 feet per 1″ per 1″ to a mass of 10 tons is obtained from the proportion of Art. 90 (2),

As 10 tons : P tons :: 32 acceleration : 4 acceleration

$\therefore P = 1\frac{1}{4}$ ton.

$\therefore$ whole tension $= 5 + 1\frac{1}{4} = 6\frac{1}{4}$ tons.

15. By Art. 99, $T = \dfrac{12 \times 9(\frac{1}{2} + \frac{1}{3})}{12 + 9} = \dfrac{108 \times \frac{5}{6}}{21} = 4\frac{2}{7}$ ozs.

16. By Art. 99, $T = \dfrac{20 \times 15(\frac{1}{2} + \frac{1}{2})}{20 + 15} = \dfrac{300}{35} = 8\frac{4}{7}$ ozs.

17. By Art. 99, $T = \dfrac{PQ(\sin i + \sin i')}{P + Q}$. If the planes become vertical the arrangement becomes the same as that of Atwood's Machine, and the inclinations i and i' become each 90°. Hence as $\sin 90^\circ = 1$, the above expression becomes,

$T = \dfrac{PQ(1 + 1)}{P + Q} = \dfrac{2PQ}{P + Q}$, which is the expression for the tension in Atwood's Machine.

If one of the planes become vertical and the other horizontal, the inclinations are 90° and 0° respectively, and therefore the expression of Art. 99 becomes,

$T = \frac{PQ(1+0)}{P+Q} = \frac{PQ}{P+Q}$, which is the expression for the tension in the arrangement of Art. 98.

18. By Art. 98, $T = \frac{PQ}{P+Q} = \frac{\frac{1}{16} \times 1}{\frac{1}{16} + 1} = \frac{1}{17}$ lb.

19. See Solution to Exercise 7. Determining the accelerations in the case of each body as in Arts. 98 and 99, and equalizing, we have—

$$\frac{36 - T}{\frac{36}{g}} = \frac{T - 36 \sin 30^\circ}{\frac{36}{g}} \therefore 36 - T = T - 36 \times \frac{1}{2} \therefore T = 27 \text{ ozs.}$$

20. Proceeding as in last solution, we have

$$\frac{Q - T}{\frac{Q}{g}} = \frac{T - P \sin i}{\frac{P}{g}} \therefore PQ - PT = QT - PQ \sin i \therefore T = \frac{PQ(\sin i + 1)}{P+Q}.$$

21. If in Art. 99 we suppose one of the planes to become vertical the arrangement will be same as that of the preceding exercise. Let $i' = 90^\circ$, then the expression of Art. 99 becomes $T = \frac{PQ(\sin i + 1)}{P+Q}$, which is that obtained in the foregoing exercise.

CHAPTER VIII.

EXAMINATION QUESTIONS.

1. Define Impact. Give examples of impacts.

2. How do impulsive forces differ from ordinary forces?

3. How are impulsive forces measured?

4. Give an account of the actions which take place when:—(1) two imperfectly elastic balls meet each other; (2) when a ball impinges on a fixed plane, the elasticity being imperfect.

5. How may the ratio of the impulsive force which acts during the first part of the impact to that which acts during the second part be determined?

6. What is the *coefficient of elasticity* for two bodies? What other names are given to this ratio?

7. What are *perfectly elastic*, *imperfectly elastic*, and *perfectly inelastic* bodies respectively?

8. A body impinges with a velocity u upon a fixed plane, and recoils with a velocity v, the coefficient of elasticity being e: state the equation connecting v and u, and show how it is obtained from Art. 110.

9. Two bodies with velocities u and u' respectively come into collision; their velocities after impact are v and v' respectively, e being the coefficient of elasticity: State the equation connecting the velocities before and after impact, and show from Art. 110 how it is obtained.

10. What is Direct Collision? What is Oblique Collision?

11. Given the masses of two spheres, their coefficient of elasticity, and their velocities before coming into direct collision, determine the velocities after collision, (1) when the spheres are moving in the same direction before impact; (2), when moving in opposite directions.

12. Show that if the bodies mentioned in the preceding question be equal and their elasticity perfect, they will interchange velocities by the collision.

13. Determine what the expressions obtained in answer to question 11 become, when the bodies are perfectly inelastic.

14. Obtain from first principles the expression for the velocity in the last case.

15. Given the masses of two spheres, their coefficient of elasticity, their velocities before coming into oblique collision, and the angles which the directions of their velocities make respectively with the line of impact, determine the values of their velocities after collision resolved in the line of impact respectively.

16. Show how the directions of the velocities after impact are determined.

17. Show how the values of the velocities after impact are determined.

18. The velocity of a body impinging directly on a plane and the coefficient of elasticity being given, determine the velocity and its direction after impact.

19. The velocity and direction of motion of a body impinging obliquely on a plane and the coefficient of elasticity being given, show how the velocity after impact and its direction may be determined.

20. In the last case determine what will be the results, if (1), the elasticity be perfect; (2), if the bodies be perfectly inelastic.

INTRODUCTORY EXAMPLES TO CHAPTER VIII.

(*a*) A body impinges directly on a fixed plane with a velocity of 24 feet per second, and recoils with a velocity of 8 feet per second: what is the coefficient of elasticity?

From Art. 110 or 112, $e=\frac{v}{u}=\frac{8}{24}=\frac{1}{3}=$ coefficient of elasticity.

(*b*) **A perfectly inelastic body having a velocity u strikes directly another perfectly elastic body of equal mass at rest: find the velocity of the former after impact.**

From Art. 115, $v=\dfrac{mu+m'u'-em'(u-u')}{m+m'}$.

And since $m=m'$, $u'=0$, and $e=1$ ∴

$$v=\frac{mu+0-mu+0}{m+m}=0.$$

(*c*) **A perfectly inelastic body whose mass is 12 and velocity 20, strikes directly another perfectly inelastic body whose mass is 8 and velocity 6: what is the velocity after impact?**

By Art. 115 or 116, $v=\dfrac{mu+m'u'}{m+m'}=\dfrac{12\times 20+8\times 6}{12+8}$
$=14\cdot 4$.

SOLUTIONS OF EXERCISES TO CHAP. VIII. Page 84.

1. By Art. 115 or 116, $v=v'=\dfrac{mu-m'u'}{m+m'}$. And since in this case $m=m'$, ∴ $v=\dfrac{m(u-u')}{2m}=\dfrac{u-u'}{2}=\dfrac{40-32}{2}=4$ feet per second.

2. By Art. 115, and since $u'=0$, and $m=m'$ ∴

$$v=v'=\frac{mu}{m+m}=\frac{u}{2}=\frac{100}{2}=50 \text{ feet per second.}$$

3. By Art. 115, $v=v'=\dfrac{mu+m'u'}{m+m'}$. And since $u'=0$, and $m'=4m$ ∴ $v=v'=\dfrac{mu}{m+4m}=\dfrac{m\times 100}{5m}=\dfrac{100}{5}=20$ feet per second.

4. By Art. 115 or 116, $v = v' = \frac{mu + m'u'}{m + m'}$. And since $u' = 0$, and $m = m'$, the velocity after the first ball impinges on the second $= v = v' = \frac{mu}{m + m} = \frac{m \times 20}{2m} = \frac{20}{2}$ feet per second.

The body $2m$ moving with the velocity $\frac{20}{2}$ feet per 1″ now impinges on the third ball of mass m. Using the same formula the velocity after impact $= \frac{2m \times \frac{20}{2}}{2m + m} = \frac{20m}{3m} = \frac{20}{3}$ feet per second.

The body $3m$ now impinges on the fourth ball, and proceeding in the same way we find the velocity after impact $= \frac{20}{4}$ feet per 1″.

Similarly the succeeding velocities are found to be $\frac{20}{5}, \frac{20}{6}, \frac{20}{7}$, &c.

5. By Art. 115,

$$v = \frac{mu + m'u' - em'(u - u')}{m + m'} = \frac{20 \times 12 + 60 \times 4 - \frac{1}{4} \times 60(12 - 4)}{20 + 60}$$

$= 4\frac{1}{2}$ feet per second.

$$v' = \frac{mu + m'u' + em(u - u')}{m + m'} = \frac{20 \times 12 + 60 \times 4 + \frac{1}{4} \times 20(12 - 4)}{20 + 60}$$

$= 6\frac{1}{2}$ feet per second.

6. By Art. 115,

$$v = \frac{mu - m'u' - em'(u + u')}{m + m'}$$

$$= \frac{15 \times 20 - 60 \times 20 - \frac{1}{4} \times 60(20 + 20)}{15 + 60} = -20 \text{ feet per second.}$$

$$v' = \frac{mu - m'u' + em(u + u')}{m + m'}$$

$$= \frac{15 \times 20 - 60 \times 20 + \frac{1}{4} \times 15(20 + 20)}{15 + 60} = -10 \text{ feet per second.}$$

7. (*a*) By Art. 115 or 116,

$$v = v' = \frac{mu - m'u'}{m + m'} = \frac{12 \times 4}{12 + 16} = 0.$$

(*b*) By Art. 115,

$$v = \frac{mu - m'u' - em'(u + u')}{m + m'} = \frac{12 \times 4 - 3 \times 16 - 3(4 + 16)}{12 + 3}$$

$= -4$ feet per second.

$$v' = \frac{mu - m'u' + em(u + u')}{m + m'} = \frac{12 \times 4 - 3 \times 16 + 12(4 + 16)}{12 + 3}$$

$= 16$ feet per second.

8. By Art. 115,

$v = \dfrac{mu + m'u' - em'(u - u')}{m + m'}$. By the question the velocity after impact becomes $\frac{3}{4}$ths of the velocity before impact $= \frac{3}{4}u$. Also $m = 4m'$, and $u' = 0$ ∴

$$\tfrac{3}{4}u = \frac{4m'u - em'u}{4m' + m'} = \frac{4u - eu}{5} \therefore 15 = 16 - 4e \therefore e = \tfrac{1}{4}.$$

9. By Art. 115, $v = \dfrac{mu + m'u' - em'(u - u')}{m + m'}$.

Since by question $m = m'$, $e = 1$, $u = 25$, and $u' = 15$ we have by substituting and dividing numerator and denominator by m,

$$v = \frac{25 + 15 - (25 - 15)}{2} = 15 \text{ feet per second.}$$

Similarly

$$v' = \frac{mu + m'u' + em(u - u')}{m + m'} = \frac{25 + 15 + (25 - 15)}{2} = 25 \text{ ft.}$$

per second.

10. In Exam. 3, p. 83 (Text-Book), it is shown that if a body fall from a height h the coefficient of elasticity being e,

the height of rebound is e^2h. Similarly if the coefficient of elasticity be e', and a body fall from the same height, the height of rebound will be e'^2h. Therefore the heights of rebound are proportional to the squares of the coefficients of elasticity; and conversely the coefficients of elasticity vary as the square roots of the heights of rebound. Hence as $\sqrt{16} : \sqrt{9} :: \frac{1}{2} : \frac{3}{8}$, which is the required coefficient of elasticity.

11. By Art. 115, $v = \dfrac{mu + m'u' - em'(u - u')}{m + m'}$; and since $m = m'$, $u' = 0$, and $e = \frac{1}{2} \therefore v = \dfrac{mu - \frac{1}{2}mu}{2m} = 2\frac{1}{2}$ feet per second.

Also Art. 115,

$$v' = \frac{mu + m'u' + em(u - u')}{m + m'} = \frac{mu + \frac{1}{2}mu}{m + m} = 7\tfrac{1}{2} \text{ feet per second.}$$

12. In Art. 115 it is shown that, when balls perfectly elastic and equal impinge, they interchange velocities. Hence when the first ball impinges on the second the first is brought to rest, and the second moves on with the velocity of 20 feet per second. Similarly with the third and fourth, so that the last moves off with the velocity of 20 feet per second, all the others being then at rest.

13. See Art. 117. Since $u' = 0$, $m = m'$, and $e = 1$, we find by substituting in equation (3), $v \cos r = \dfrac{mu \cos i - mu \cos i}{2m} = 0$.

Dividing equation (1) by (3), we have $\dfrac{v \sin r}{v \cos r} = \dfrac{u \sin i}{0}$.

Therefore $\tan r = \infty \therefore r = 90^\circ$ (see Fig. to Art. 117).

14. As in the last solution by substituting in equations (2) and (4) and dividing, we have $\tan r' = 0 \therefore r' = 0^\circ$. (See Fig. to Art. 117.)

15. In Exam. (3), p. 83, Text Book, it is shown that when a ball falls from a height h the height to which it rebounds $= e^2h$ ∴

$$\text{Height} = e^2h = \frac{1}{16} \times 48 = 3 \text{ feet.}$$

16. See Exam. (3), p. 83, Text Book.

Height of rebound = height fallen × e^2. Therefore

$$\text{Height of 1st rebound} = e^2h = \frac{1}{4} \times 96 = 24 \text{ feet.}$$

$$\text{Height of 2nd rebound} = \frac{1}{4} \times 24 = 6 \text{ feet.}$$

$$\text{Height of 3rd rebound} = \frac{1}{4} \times 6 = 1\tfrac{1}{2} \text{ feet.}$$

$$\text{Height of 4th rebound} = \frac{1}{4} \times 1\tfrac{1}{2} = \frac{3}{8} \text{ foot.}$$

17. We employ equation (3) of Art. 117 to find resolved velocity after impact. And since by the question $m = m'$, $u' = 0$, and $e = 1$, substituting in equation (3), we find as in Solution to Exercise 13, $v \cos r = 0$. Dividing this equation into equation (1), Art. 117, we have $\frac{v \sin r}{v \cos r} = \frac{u \sin i}{0}$. Therefore $\tan r = \infty$ $\therefore r = 90°$.

But by question (and see fig. of Art. 117), $r - i = 60°$
$\therefore i = r - 60° = 30°$.

Squaring equations (1) and (3) of Art. 117 and adding, we have

$$v^2 \sin^2 r = u^2 \sin^2 i$$

$$v^2 \cos^2 r = 0$$

$$\therefore v^2(\sin^2 r + \cos^2 r) = u^2 \sin^2 i.$$

But by trigonometry, $\sin^2 r + \cos^2 r = 1$, and by question $u = 50$, and $i = 30^\circ$

$\therefore v^2 = 50^2 \sin^2 30^\circ$

$\therefore v = 50 \sin 30^\circ = 50 \times \frac{1}{2} = 25$ feet per second.

18. Let i = angle of incidence, and r = angle of reflection; then, Art. 119, $\tan r = \frac{1}{e} \tan i = 3 \tan i \left(\text{since } e = \frac{1}{3} \right)$.

By question $i + r = 90^\circ \therefore r = 90^\circ - i \therefore \tan (90^\circ - i) = 3 \tan i$.

But by trigonometry, $\tan (90^\circ - i) = \cot i = \frac{1}{\tan i} \therefore 3 \tan i = \frac{1}{\tan i} \therefore 3 \tan^2 i = 1 \therefore \tan^2 i = \frac{1}{3}$, and $\tan i = \frac{1}{\sqrt{3}}$ $= \tan 30^\circ$. Therefore $i = 30^\circ$.

19. Let u = velocity of projection, e being $\frac{1}{2}$. We employ equation (6), Art. 14, to find the velocity after passing through a given space, $v^2 = u^2 \pm 2gs$. Therefore

Velocity *just* before striking the ceiling $= \sqrt{u^2 - 2 \times g \times 12}$ $= \sqrt{u^2 - 24g}$.

Velocity *just* after rebounding from ceiling (Art. 112) $= e\sqrt{u^2 - 24g}$.

This forms the initial velocity with which the body descends; and the velocity after passing through any space downwards is obtained from equation (6), Art. 14, using + sign. Therefore

Velocity *just* before striking the floor

$$= \sqrt{(e\sqrt{u^2 - 24g})^2 + 2gs} = \sqrt{e^2(u^2 - 24g) + 24g}\,.$$

And (Art. 112) velocity *just* after rebounding from the floor $= e\sqrt{e^2(u^2 - 24g) + 24g}$. With this initial velocity the ball just reaches to the height of 12 feet, that is when it reaches this height its velocity becomes 0. Therefore using the general equation, $v^2=u^2-2gs$, and substituting we have

$$0=(e\sqrt{e^2(u^2-24g)+24g})^2-2\times g\times 12$$

$$\therefore (e\sqrt{e^2(u^2-24g)+24g})^2=24g$$

$$\therefore e\sqrt{e^2(u^2-24g)+24g}=\sqrt{24g}$$

$$\therefore \sqrt{e^2(u^2-24g)+24g}=2\sqrt{24g}$$

$$\therefore e^2(u^2-24g)+24g=96g$$

$\therefore e^2(u^2-24g)=72g \therefore u^2=312g \therefore u=100$ feet per second nearly.

20. Since the velocities are in opposite directions the equation of Art. 115 becomes, $v=\dfrac{mu-m'u'-em'(u+u')}{m+m'}$

And by question $m=3m'$, $e=1$, and $u=u'$ $\therefore$ substituting

$$v=\frac{3m'u-m'u-m'u-m'u}{3m'+m'}=0.$$

CHAPTER IX.

EXAMINATION QUESTIONS.

1. What is the measure of the effect of a force acting on a given mass for a given time?

2. Define Work. When is work done by a force, and when against a force?

3. Give examples of the production of work.

4. Define Energy. How is energy divided?

5. Explain what is meant by Potential and what by Kinetic energy, and give illustrations of each.

6. Give examples of the conversion of one form of energy into another.

7. Investigate an expression for the Statical measure of work, and one for the Kinetical measure.

8. What is the measure of the energy of a moving body?

9. Show how to determine the space through which a body possessing a given amount of kinetic energy will move against a given resistance.

10. How is the work done determined in any case where the motion is not in the line of action of the force?

11. Show that there is no work done when the motion is at right angles to the direction of the force.

12. Examine the two equations, $Ft = mv$, and $Fs = \frac{1}{2}mv^2$.

13. What is meant by *vis viva?*

14. State the Principle of Conservation of Energy, and show how it has been obtained.

15. State the Principle of Work, and show how it has been obtained.

16. Show that the Principle of the Conservation of Energy may be deduced from Newton's Third Law of Motion.

17. Give the equation which expresses the Principle of Work in the case of any machine, (1) friction not being taken into account, (2) with friction.

18. Show from the Principle of Work that the problem of "perpetual motion" is impossible.

19. What is meant by the *modulus* of a machine?

20. What are the gravitation units of work usually adopted in this country?

21. What is the metrical gravitation unit of work?

22. What are the absolute units of work that are adopted?

23. How may gravitation units of work be reduced to kinetic units and conversely?

INTRODUCTORY EXAMPLES TO CHAPTER IX.

(*a*) A mass of stone weighing 2 cwts. rests on the top of a vertical precipice 30 yards high: what is its potential energy?

By Art. 124, the potential energy $= Wh = (2 \times 112) \times (30 \times 3) = 224 \times 90 = 20{,}160$ foot-pounds.

(*b*) What will be the kinetic energy of the body at the moment it reaches the ground?

By Art. 125, the kinetic energy $= \frac{1}{2}mv^2$. The mass of the body expressed in gravitation units (see Art. 80) $= \frac{2 \times 112}{32}$. The velocity it acquires in falling through 90 feet is obtained by equation (3), Art. 14, $v^2 = 2gs = 2 \times 32 \times 90 \therefore v = \sqrt{2 \times 32 \times 90}$ feet per second. The kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times \frac{2 \times 112}{32} \times (2 \times 32 \times 90) = 20{,}160$ foot-pounds.

NOTE.—Examples (*a*) and (*b*) illustrate the statements of Art. 123.

(*c*) The weights in Atwood's Machine are 4 lbs. and 8 lbs. respectively, and the system is so arranged that the former weight falls freely through a height of 4 feet before the heavier weight is pulled by the string: how high will the latter weight be raised before the system comes to rest?

The 4 lbs. weight in falling through 4 feet acquires a velocity which is obtained from the equation, $v^2 = 2gs$ $\therefore v = \sqrt{2 \times 32 \times 4} = 16$ feet per second. The resisting force which tends to bring the system to rest $= 8 - 4 = 4$ lbs. The mass in motion $= 4$ lbs., and this mass expressed in gravitation units $= \frac{4}{32}$ lb. (Art. 80). The space through which the system moves is obtained from equation of Art. 127.

$$s = \frac{\text{kinetic energy}}{\text{resistance}} = \frac{\frac{1}{2}mv^2}{R} = \frac{\frac{1}{2} \times \frac{4}{32} \times 16^2}{4} = 4 \text{ feet.}$$

NOTE.—We have in this example an illustration of the statement made in Art. 126. If the second weight of the system mentioned were 4 lbs. then when this weight would be acted on by the string the whole system would move uniformly with the velocity that the first 4 lbs. weight had acquired, that is 16 feet per second (friction being neglected). 4 lbs. weight of the 8 lbs. acts in this way to counterbalance the 4 lbs. weight that has been in motion, so that the whole kinetic energy acquired by the latter is expended in raising an equal weight, and it raises it to the height from which itself fell.

(*d*) A body weighing 100 lbs. is pushed up a smooth inclined plane whose height is 5 feet and length 20 feet: What is the work done?

See Art. 128. The body whose weight is 100 lbs. acting vertically downwards is raised through the vertical height of 5 feet $\therefore$ the work done $= Wh = 100 \times 5 = 500$ foot-pounds.

Or :—Resolving the 100 lbs. weight along the inclined plane we have the amount of the force acting along the plane $= 100 \times \frac{5}{20} = 25$ lbs. This is pushed through a space of 20 feet $\therefore$ the work done against this force $= 25 \times 20 = 500$ foot-pounds.

(*e*) What work can be done in an hour by a steam engine of 5 Horse power?

Art. 137. A Horse power = 33,000 lbs. raised 1 foot high per minute ∴ 5 H.P. = 5 × 33000 foot-pounds per minute = 5 × 33000 × 60 foot-pounds per hour = 9,900,000 foot-pounds.

SOLUTIONS OF EXERCISES TO CHAPTER IX. PAGE 96.

1. By Art. 124, the work done = Wh. And $W = 12 \times 14$ pounds, or $12 \times 14 \times g$ poundals (see Arts. 82 and 85).

∴ Work done = $(12 \times 14) \times 40 = 6720$ foot-pounds.

And, Work done = $(12 \times 14 \times g) \times 4 = 6720g$ foot-poundals.

2. $1500 \times 2240 \times 330$ = foot-pounds of work to be done per day.

$33000 \times 10 \times 60$ foot-pounds = the work done by one Horse power in a day of 10 hours (Art. 137).

$$\therefore \frac{1500 \times 2240 \times 330}{33000 \times 10 \times 60} = \text{Horse power required} = 56 \text{ H.P.}$$

3. $660 \times 2240 \times 30$ = work to be done in foot-pounds.

30×33000 = foot-pounds per minute done by engine.

$$\therefore \frac{660 \times 2240 \times 30}{30 \times 33000} = \text{No. of minutes} = 150 \text{ minutes}$$ nearly = $2\frac{1}{2}$ hours.

4. (*a*) Art. 124, the work done = Wh. $W = 1 \times 2240$ pounds, and $h = 100 \times 6$ feet ∴ Work done = $(1 \times 2240) \times (100 \times 6) = 1,344,000$ foot-pounds.

(*b*) $W = (1 \times 2240 \times g)$ foot-poundals (see Arts. 82 and 85), and $h = 100 \times 6$ feet ∴ Work done = $(1 \times 2240 \times g) \times (100 \times 6) = 1,344,000g$ foot-poundals.

5. $60 \times 33000 \times 10 \times 60 =$ foot-pounds done per day of 10 hours.

$\therefore \dfrac{60 \times 33000 \times 10 \times 60}{10} =$ gallons raised 1 foot high per day.

$\therefore \dfrac{60 \times 33000 \times 10 \times 60}{10 \times 50 \times 60} =$ gallons raised 50 fathoms high per day $= 396{,}000$ gallons.

6. $2 \times 25 \times 33000 \times 60 =$ foot-pounds of work which the engines can do per hour.

$\therefore \dfrac{2 \times 25 \times 33000 \times 60}{100 \times 6} =$ number of pounds raised from a depth of 100 fathoms per hour by the engines.

$\therefore \dfrac{2 \times 25 \times 33000 \times 60}{100 \times 6 \times 62\frac{1}{2}} =$ number of cubic feet of water raised $= 2640$ cubic feet.

7. The cubic content $= 10 \times 30 = 300$ cubic feet; and the weight $= 300 \times 100 = 30{,}000$ lbs. This weight is raised from different depths, but the work done is the same as if the whole were raised from the depth of the centre of gravity of the mass, which is 15 feet.

$\therefore 30{,}000 \times 15 = 450{,}000$ foot-pounds.

8. The potential energy of the water is the work it can perform $= Wh =$ cubic content in feet $\times 62\frac{1}{2}$ (weight of cubic foot of water in lbs.) $\times 4 \therefore$

$(120 \times 50 \times 2) \times 62\frac{1}{2} \times 4 = 3{,}000{,}000$ foot-pounds.

9. By Art. 125 the kinetic energy or kinetic measure of the work done $= \frac{1}{2}mv^2$. The cubic foot of water weighs $62\frac{1}{2}$ lbs., and its mass expressed in gravitation units $= \dfrac{62\frac{1}{2}}{g}$. (See Art. 80.) Therefore kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times \dfrac{62\frac{1}{2}}{g} \times g^2$ $= 31 \cdot 25g$ foot-pounds.

10. 50 H.P. less 25 per cent. $= 50 - 12\frac{1}{2} = 37\frac{1}{2}$ H.P.

$\therefore \dfrac{37\frac{1}{2} \times 33000 \times 60}{410 \times 62\frac{1}{2}} =$ cubic feet required $= 2897 \cdot 5$ cubic feet. See solutions to Exercises 5 and 6.

11. $\dfrac{900000}{9 \times 60} = \dfrac{10000}{6} =$ units of work done per minute.

1 Horse-power $= 33000$ units work per minute.

$\therefore \dfrac{\frac{10000}{6}}{33000} =$ fraction required $= \dfrac{5}{99}$ of H. P.

12. $10 \times 2240 \times 600 =$ foot-pounds of work to be done per hour.

$33000 \times 60 =$ foot-pounds done per hour by 1 H. P.

$\therefore \dfrac{10 \times 2240 \times 600}{33000 \times 60} =$ H. P. required $= 6\frac{26}{33}$ H. P.

13. See Art. 136. $3000 \times 62\frac{1}{2} \times 30 =$ work actually done per hour in foot-pounds. $15 \times 33000 \times 60 =$ theoretical work in foot-pounds per hour $\therefore$

$\dfrac{3000 \times 62\frac{1}{2} \times 30}{15 \times 33000 \times 60} =$ modulus $= \dfrac{25}{132}$.

14. Both balls being thrown with the same velocity, the lead ball, since it contains a greater mass, will (Art. 125) possess a greater energy than the cork, and hence (Art. 127) it will move through a greater distance than the cork against the same opposing resistance of the air.

15. We first find the acceleration by Art. 90 (3).

$$f = \frac{P}{P + Q} g = \frac{1}{32} \times 32 = 1 \text{ foot per sec. per sec.}$$

Again by equation (1), Art. 12, $v = ft = 1 \times 3 = 3$ feet per sec. the velocity when the string broke.

The body now possessing a velocity of 3 feet per 1″ moves against a resisting force through a space of $4\frac{1}{2}$ feet. To find this resistance. By Art. 127,

$$\text{space} = \frac{\text{kinetic energy}}{\text{resistance}} \therefore \text{resistance} = \frac{\text{kinetic energy}}{\text{space}} = \frac{\frac{1}{2}mv^2}{s} = \frac{\frac{1}{2} \times \frac{31}{32} \times 3^2}{4\frac{1}{2}} = \frac{31}{32} \text{ lb.}$$

NOTE.—The mass of 31 lbs. is expressed in gravitation units of mass by dividing by 32. See Art. 80.

16. By Art. 127, $s = \frac{\frac{1}{2}mv^2}{R} \therefore R = \frac{\frac{1}{2}mv^2}{s} = \frac{\frac{1}{2} \times \frac{12}{32} \times 64^2}{192 \times 3}$ $= 1\frac{1}{3}$ lb.

17. See Solution to Exercise 15.

(a) $f = \frac{P}{P+Q}g = \frac{2}{2+24} \times 32 = \frac{32}{13}$ feet per 1″ per 1″.

(b) $v = ft = \frac{32}{13} \times 3 = \frac{96}{13}$ feet per 1″.

(c) By Art. 89, time $= \frac{\text{momentum}}{\text{resistance}} \therefore$ resistance $=$

$$\frac{\text{momentum}}{\text{time}} = \frac{mv}{t} = \frac{\frac{24}{32} \times \frac{96}{13}}{4} = 1\frac{5}{13} \text{ lb.}$$

18. (a) $v = ft = 4 \times 10 = 40$ feet per 1″, the velocity when the rope breaks.

By Art. 89, time $= \frac{\text{momentum}}{\text{resistance}}$. The resistance which opposes the motion is the weight of the waggon resolved along the plane $= 10 \times 2240 \times \sin 30^\circ$ $= 5 \times 2240$. The mass expressed in gravitation units $= \frac{10 \times 2240}{32}$. Therefore, Art. 89,

$$\text{time} = \frac{\text{momentum}}{\text{resistance}} = \frac{\frac{10 \times 2240}{32} \times 40}{5 \times 2240} = 2\frac{1}{2} \text{ seconds.}$$

(*b*) By Art. 127, $\text{space}=\dfrac{\text{kinetic energy}}{\text{resistance}}=\dfrac{\frac{1}{2}mv^2}{\text{R}}=$

$$\frac{\frac{1}{2}\times\dfrac{10\times2240}{32}\times40^2}{5\times2240}=50\text{ feet.}$$

19. (*a*) By Art. 90 (3), $f=\dfrac{\text{P}}{\text{P}+\text{Q}}g=\dfrac{10}{50+10}\times32=\dfrac{16}{3}$ feet per 1″ per 1″.

(*b*) Art. 12, $v=ft=\dfrac{16}{3}\times3=16$ feet per 1″.

(*c*) Art. 127, $\text{space}=\dfrac{\text{kinetic energy}}{\text{resistance}}\;\therefore\;\text{R}=\dfrac{\frac{1}{2}mv^2}{s}=$

$$\frac{\frac{1}{2}\times\frac{50}{32}\times16^2}{4\frac{1}{2}}=16\text{ lbs.}$$

20. (*a*) We find the acceleration by Art. 90 (2).

$$f=\frac{\text{P}-\text{Q}}{\text{P}+\text{Q}}g=\frac{5-4}{5+4}\times32=\frac{32}{9}\text{ feet per 1″ per 1″.}$$

(*b*) Art. 12, $v=ft=\dfrac{32}{9}\times9=32$ feet per 1″ when the bar is removed.

(*c*) The resisting force that now opposes the motion is the difference of the weights $=4-3=1$ lb.; and the mass moving = mass of $(3+4)$ lbs. $=\dfrac{3+4}{32}$ gravitation units of mass.

By Art. 127, $$s=\frac{\frac{1}{2}mv^2}{\text{R}}=\frac{\frac{1}{2}\times\dfrac{3+4}{32}\times32^2}{1}=112\text{ feet.}$$

21. See Arts. 125 and 131.

22. See Art. 137.

23. (1) By Art. 89, $t=\frac{mv}{R}=\frac{\frac{100}{32}\times 96}{8}=37\frac{1}{2}$ secs.

(2) By Art. 127, $s=\frac{\frac{1}{2}mv^2}{R}=\frac{\frac{1}{2}\times\frac{100}{32}\times 96^2}{8}=1800$ feet.

24. Friction $=\frac{1}{10}$ of 20 lbs. = 2 lbs.

(1) Art. 89, $t=\frac{mv}{R}=\frac{\frac{20}{32}\times 60}{2}=18\frac{3}{4}$ secs.

(2) Art. 127, $s=\frac{\frac{1}{2}mv^2}{R}=\frac{\frac{1}{2}\times\frac{20}{32}\times 60^2}{2}=562\frac{1}{2}$ feet.

25. By Art. 127, Resistance $=\frac{\frac{1}{2}mv^2}{s}=\frac{\frac{1}{2}\times\frac{1000}{32}\times 48^2}{200}=$ 180 lbs.

26. 15 miles per hour $=\frac{15\times 5280}{60}$ feet per minute.

$60\times 20=1200$ lbs. = resistance of air and friction.

$\frac{15\times 5280}{60}\times 1200$ = foot-pounds of work to be done per minute $\therefore \frac{\frac{15\times 5280}{60}\times 1200}{33000}$ = Horse-power required = 48 H. P.

27. Resistance in this case = resistance of air and friction + $\frac{1}{80}$th of weight of train = 1200 lbs. + 2240 lbs. = 3440 lbs. (see Art. 128).

$\therefore \frac{15\times 5280}{60}\times 3440$ = foot-pounds of work to be done per minute $\therefore \frac{\frac{15\times 5280}{60}\times 3440}{33000}$ = Horse-power = 137·6 H. P.

28. 20 miles per hour $= \frac{88}{3}$ feet per sec. Friction = 8 lbs. per ton = 8 × 200 = 1600 lbs. By Art. 127,

$$s = \frac{\frac{1}{2}mv^2}{R} = \frac{\frac{1}{2}\times\frac{200\times 2240}{32}\times\left(\frac{88}{3}\right)^2}{1600} = 3764\cdot 4 \text{ feet.}$$

29. 40 miles per hour $= \frac{176}{3}$ feet per 1″.

Friction = 8 × 60 = 480 lbs. By Art. 128, the weight lifted in ascending the plane $= \frac{1}{100}$ of 60 tons = 1344 lbs. Therefore the total resistance = 480 + 1344 = 1824 lbs. By Art. 127,

$$s = \frac{\frac{1}{2}mv^2}{R} = \frac{\frac{1}{2}\times\frac{60\times 2240}{32}\times\left(\frac{176}{3}\right)^2}{1824} = 3962 \text{ feet.}$$

30. Friction, &c., = 1400 lbs. Weight lifted $= \frac{1}{50}$ of 100 tons = 4480 lbs. ∴ total resistance = 5880 lbs. 30 miles per hour = 2640 feet per minute ∴

5880 × 2640 = foot-pounds of work to be done per minute

$$\therefore \frac{5880\times 2640}{33000} = \text{Horse-power} = 470\cdot 4 \text{ H.P.}$$

31. To find the velocity when the rope breaks.

$v = ft = 20 \times 6 = 120$ feet per 1″.

The weight resolved along the incline $= 1120 \times \frac{1}{56} = 20$ lbs.

Friction $= \frac{1120}{200} = \frac{28}{5}$ lbs. Therefore whole resistance $= \left(20 + \frac{28}{5}\right)$ lbs.

(a) By Art. 89, $t = \frac{mv}{R} = \frac{\frac{1120}{32}\times 120}{20+\frac{28}{5}} = 164\frac{1}{16}$ seconds.

(*b*) Force producing motion down the plane = resolved part of weight − friction $= 20$ lbs. $- \frac{28}{5}$ lbs. $= \frac{72}{5}$ lbs.

The acceleration that this force will produce in a mass of 10 cwts. is found from Art. 90 (1). See solution to exercise 1, Chap. VI.

As $1120 : \frac{72}{5} :: 32 : f \therefore f = \frac{72}{175}$ feet per 1″ per 1″.

The velocity obtained with this acceleration is found by Art. 12, equation (1).

$v = ft = \frac{72}{175} \times 164\frac{1}{16} = 67\frac{1}{2}$ feet per second.

32. (*a*) Art. 89, $t = \frac{mv}{R} = \frac{\frac{1120}{32} \times 120}{20} = 210$ seconds.

Or: Art. 43, and see Solution to Exercise 13, Chap. IV:—

$v = u - g \sin i\, t \therefore 0 = 120 - 32 \times \frac{1}{56} \times t \therefore t = 210$ seconds.

(*b*) Art. 43, $v = g \sin i\, t = 32 \times \frac{1}{56} \times 210 = 120$ feet per second.

33. $1 \times 2240 \times 32 =$ poundals in a ton. (Art. 82.)

$1 \times 2240 \times 32 \times 10 =$ foot-poundals of work done $= 716{,}800$ foot-poundals.

34. Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 100 \times 20^2 = 20{,}000$ foot-poundals.

The mass of 100 lbs. = 100 absolute units of mass. (See Art. 82.)

35. Work done $= Wh$. See Arts. 137 and 138. Using gravitation units, and taking the gramme as the unit of weight and the centimetre as the unit of length, 500 grammes = 500 units of weight, and 5 metres $= 5 \times 100$ units of length $\therefore$

$Wh = 500 \times 500 = 250{,}000$ centimetre-grammes.

Using absolute units, 500 grammes $= 500 \times 980$ dynes (see Art. 82), and 5 metres $= 5 \times 100$ centimetres $\therefore$

$Wh = 500 \times 980 \times 500 = 245,000,000$ centimetre-dynes or ergs.

NOTE.—It will be seen that the centimetre-grammes may be reduced to ergs by multiplying by g. (See Art. 138.)

36. 1 kilogramme = 1000 grammes, and 1 metre = 100 centimetres.

Using absolute units 1000 grammes = 1000 units of force, and 100 centimetres = 100 units of length. (See Art. 82.)

By Art. 125, Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 1000 = 100^2$ $= 5,000,000$ ergs.

37. Let v = velocity in feet per second.

Then $120 \times 10 \times v$ = work to be done per second in foot-pounds.

And $\frac{30 \times 33000}{60}$ = work done per second by engine in foot-pounds.

$\therefore 120 \times 10 \times v = \frac{30 \times 33000}{60}$ $\therefore v = \frac{55}{4}$ feet per second = $9\frac{3}{8}$ miles per hour.

CHAPTER X.

EXAMINATION QUESTIONS.

1. Show that if a body move with uniform velocity in the circumference of a circle, it must be acted on by a constant force directed towards the centre.

2. Given the mass of the body moving uniformly in the circumference of a circle $= m$, its velocity v, and the radius of the circle r: find the centripetal force.

3. What is meant by the so-called *centrifugal* force? Point out its true nature.

4. Show that no tangential force acts during the uniform circular motion.

5. Express the centripetal force in terms of the mass and the periodic time.

6. What is meant by the angular velocity of a body?

7. Express the centripetal force in terms of the mass and the angular velocity.

8. Give illustrations of centrifugal force.

9. Give an account of the effects produced by the centrifugal force due to the earth's rotation.

INTRODUCTORY EXAMPLES TO CHAPTER X.

(*a*) A body whose weight is 8 lbs. attached to a string 4 feet long, is moving uniformly in a circular path with a velocity of 20 feet per second: what is the tension of the string?

By Art. 140, the tenison $= \mathrm{F} = \frac{mv^2}{r}$. Since the weight is 8 lbs., there are $\frac{8}{32}$ gravitation units of mass in the body $\therefore \mathrm{F} = \frac{mv^2}{r} = \frac{\frac{8}{32} \times 20^2}{4} = 25$ lbs.

(*b*) In the foregoing case find the periodic time.

Art. 144, $T = \frac{2\pi r}{v} = \frac{2 \times 3\cdot14159 \times 4}{20} = 1\cdot2566$ seconds.

(*c*) A body weighing 16 lbs. is moving uniformly in a circle whose radius is 24 feet, and makes a revolution every 8 seconds : what is the centripetal force?

By Art. 144, $F = m\frac{4\pi^2 r}{T^2} = \frac{16}{32} \times \frac{4 \times \pi^2 \times 24}{8^2} = \frac{3}{4}\pi^2 = \frac{3}{4}(3\cdot14159)^2$ lbs.

SOLUTIONS OF EXERCISES TO CHAPTER X. PAGE 104.

1. By Art, 140, $f = \frac{v^2}{r}$, where f is the acceleration corresponding to the centripetal force, or to the reaction called the centrifugal force. If there were no force of gravity, the string would always have a tension equal to this force. But if when the stone is vertically above the hand, the string *just* remains stretched without tension, the force of gravity acting on the stone must be exactly equal to the tension, and the acceleration due to gravity g must be equal and opposite to f $\therefore$

$$g = \frac{v^2}{r} \therefore v^2 = rg \therefore v = \sqrt{4g} = 2\sqrt{g} \text{ feet per second.}$$

2. 3 miles per hour $= \frac{3 \times 5280}{60 \times 60}$ feet per second; and 1 ton $= \frac{2240}{32}$ gravitation units of mass.

By Art. 140, $F = \frac{mv^2}{r} = \frac{\frac{2240}{32} \times \left(\frac{3 \times 5280}{60 \times 60}\right)^2}{132 \times 3} = 3\cdot4$ lbs.

3. By Art. 144, $F = \frac{m4\pi^2 r}{T^2} = \frac{\frac{4}{32} \times 4 \times \pi^2 \times 2}{3^2} = \frac{\pi^2}{9}$ lbs.

4. By Art. 140, $F = \frac{mv^2}{r} \therefore 36 = \frac{\frac{4}{32} \times v^2}{2} \therefore v = \sqrt{\frac{36 \times 2 \times 32}{4}}$ $= 24$ feet per second.

5. By Art. 140, $F = \dfrac{mv^2}{r} = \dfrac{\dfrac{12}{32} \times \left(\dfrac{5280 \times 30}{60 \times 60}\right)^2}{500 \times 3} = \cdot 484$ lbs.

NOTE.—The miles per hour are reduced to feet per second, and the mass is expressed in gravitation units.

6. By Art. 144, $F = \dfrac{m 4\pi^2 r}{T^2} = \dfrac{\frac{8}{32} \times 4\pi^2 \times 10}{4^2} = \dfrac{5}{8}\pi^2$ lbs.

7. See Art. 140. There must be a force acting on the body, and this force must be continually directed to the centre of the circular path.

8. See Art. 142.

9. See Arts. 147 and 148. (1) At the poles no effect is produced. (2) At the equator a body tends to fly off in a tangent to the equator. The observed weight of the body and the acceleration due to gravity are less than they would be if there were no rotation. (3) At an intermediate place (*a*), the weight of the body and the acceleration due to gravity are less than they would be if there were no rotation, but are not diminished so much as at the equator; (*b*), the body tends to move towards the equator.

10. See Art. 147, where the proof is given.

11. The stone being whirled horizontally, gravity does not affect the tension. By Art. 140,

$$F = \frac{mv^2}{r} \therefore 2 = \frac{\frac{1}{32} \times v^2}{4} \therefore v = 16 \text{ feet per second.}$$

12. $F = \dfrac{mv^2}{r} = \dfrac{\frac{5}{32} \times 80^2}{10} = 100$ lbs. This force is directed towards the centre by Art. 140.

13. 15 miles per hour $= 22$ feet per second, and 10 tons $= \dfrac{10 \times 2240}{32}$ gravitation units of mass. By Art. 140, F

$$= \frac{mv^2}{r} = \frac{\dfrac{10 \times 2240}{32} \times 22^2}{1000} = 338 \cdot 8 \text{ lbs.}$$

CHAPTER XI.

EXAMINATION QUESTIONS.

1. Show that the velocity acquired by a body in falling down a circular arc in a vertical plane is equal to what it would acquire in falling through the vertical height of the arc.

2. Show how the velocity acquired in falling down an arc of a circle in a vertical plane may be expressed in terms of the chords drawn from the extremities of the arc to the lowest point of the circle.

3. What is a simple pendulum? Give an account of its motion. Define the terms *length of pendulum*, *amplitude of vibration*, *time of vibration*.

4. What is a compound pendulum?

5. [*] Investigate the formula which expresses the time of vibration of a simple pendulum.

6. State the relations which subsist between the time of vibration, the length of the pendulum, and the acceleration of gravity at any place.

7. How may it be inferred that the earth attracts all kinds of matter alike?

8. How may the formula which expresses the time of vibration be verified by experiment?

9. Show that with different pendulums in the same place the lengths are directly proportional to the squares of the times of vibration.

10. Show that with the same pendulum in different places the accelerations produced by gravity are inversely proportional to the squares of the times of vibration, and directly proportional to the squares of the numbers of vibrations in the same time.

11. What are the limitations to the truth of the formula which expresses the time of vibration of a pendulum?

12. How may the formula be extended to cases outside these limits?

13. Define the terms *centre of oscillation*, and *centre of suspension*?

14. What is meant by the *length* of a compound pendulum?

15. Describe methods by which the length of a compound pendulum may be experimentally determined?

16. Explain how the time of vibration of a pendulum may be determined experimentally with great accuracy.

17. What is the most accurate method of determining the acceleration due to gravity at any place? Show how the value of g is obtained.

18. Explain why the value of g varies with the latitude.

19. How does the value of g vary with the altitude?

20. Given the value of g at any place, show how the length of the seconds pendulum at that place may be calculated.

INTRODUCTORY EXAMPLES TO CHAP. XI.

(*a*) A pendulum 6 inches long is vibrating in a place where $g = 32$: what is the time of vibration?

By Art. 154, $t = \pi\sqrt{\frac{l}{g}}$, where l and g must be expressed in the same units of length, and since $g = 32$ *feet* per 1″ per 1″, $l = \frac{6}{12} = \frac{1}{2}$ foot. Therefore $t = \pi\sqrt{\frac{l}{g}}$ $= 3\cdot14159\sqrt{\frac{\frac{1}{2}}{32}} = 3\cdot14159 \times \frac{1}{8} = 0\cdot3927$ second.

(*b*) What is the length of the seconds pendulum in a place where $g = 32\cdot2$?

By Art. 154, $t = \pi\sqrt{\frac{l}{g}} \therefore 1 = 3\cdot14159\sqrt{\frac{l}{32\cdot2}} \therefore 1^2$ $= 3\cdot14159^2 \times \frac{l}{32\cdot2} \therefore l = \frac{32\cdot2}{3\cdot14159^2} = 3\cdot2625$ feet $=$ $39\cdot15$ inches.

SOLUTIONS OF EXERCISES TO CHAPTER XI. PAGE 115.

1. By Art. 154, $t = \pi\sqrt{\frac{l}{g}}$, where $t = 1$ second. Therefore $1 = \pi\sqrt{\frac{l}{32 \cdot 19}}$ $\therefore 1^2 = \pi^2 \times \frac{l}{32 \cdot 19}$ $\therefore l = \frac{32 \cdot 19}{\pi^2} = \frac{32 \cdot 19}{3 \cdot 14159^2} = 39 \cdot 1393$ inches.

2. By equation of Art. 154, or by Art. 157, the lengths are proportional to the squares of the times of vibration $\therefore$

$1^2 : (\frac{1}{4})^2 :: 39 \cdot 1393 : 2 \cdot 446$ inches.

3. The number of oscillations made at the equator in the day $= 86400$, and at the poles $= 86400 + 300 = 86700$. From equation of Art. 154, the acceleration due to gravity is inversely proportional to the square of the time of oscillation, and therefore directly proportional to the square of the number of oscillations in a given time $\therefore$

> Force of gravity at equator : Force at poles :: $86400^2 : 86700^2$, or as $144 : 145$ nearly.

4. $39 \cdot 1393 - \frac{1}{20} = 39 \cdot 0893$ inches. From Art. 154 the length is inversely proportional to the square of the number of oscillations in a given time. Hence

> As $39 \cdot 0893 : 39 \cdot 1393 :: 86400^2 : x^2$, where x is the number of oscillations made by the shortened pendulum in a day. From this proportion $x = 86457 \cdot 7$ secs. Then $x - 86400 = 57 \cdot 7$ secs.

5. By Art. 163, as $(4000 + 3)^2 : 4000^2 :: g : g'$. And by the equation of Art. 154, the accelerations at the base and at the top are directly proportional to the squares of the numbers of oscillations made in a day. The number of oscillations made in a day at the base $= 86400$. Let $x =$ number made at the top, then as

> $4003^2 : 4000^2 :: 86400^2 : x^2$, from which x is found to be $86335 \cdot 2$. Then $86400 - 86335 \cdot 2 = 64 \cdot 8$ seconds.

6. By Art 154 the lengths are directly proportional to the accelerations due to gravity. Therefore the forces of gravity in the two places are as $39{\cdot}15 : 39{\cdot}14$, or as $1 : {\cdot}9997$.

7. From equation of Art. 154, the lengths are directly proportional to the squares of the times of vibration. Therefore if l' be the length of the pendulum that would vibrate once in 60 seconds, we have

$l' : 39{\cdot}15 :: 60^2 : 1^2 \therefore\ l' = 39{\cdot}15 \times 3600$ inches $= 11{,}745$ feet.

8. From equation of Art. 154, the lengths are directly proportional to the accelerations of gravity. Let $x =$ length of pendulum in Dublin, then

As $32{\cdot}19 : 32{\cdot}2 :: 39{\cdot}1393 : x \therefore x = 39{\cdot}151$ inches.

9. See Solution to Exercise 3. Let $x =$ the number of vibrations made in a day by the pendulum when brought to the equator, the number made in London being 86,400. By equation of Art. 154 the accelerations are directly proportional to the squares of the numbers of vibrations. Therefore as

$32{\cdot}19 : 32{\cdot}09 :: 86400^2 : x^2$, from which x is found to be 86265. Then $86400 - 86265 = 135$ seconds, the number lost at the equator.

CHAPTER XII.

EXAMINATION QUESTIONS.

1. Define Statics.

2. When it is said that the forces acting on a body equilibrate each other what is meant?

3. In problems in Statics why is it not necessary to take into account the mass of the body on which the forces act?

4. Define Composition of Forces. How is the problem divided?

5. Show how to obtain the resultant of any number of forces acting in the same straight line.

6. State and prove the Parallelogram of Forces.

7. Show how the resultant of two forces may be calculated in certain special cases without the aid of trigonometry.

8. Determine the general formula for calculating the resultant of two forces.

9. How may the parallelogram of forces be verified experimentally?

10. State and prove the Triangle of Forces.

11. Show that if three forces acting on a particle be in equilibrium, each force is proportional to the sine of the angle between the directions of the other two.

12. State and prove the Polygon of Forces.

13. State and prove the Parallelopiped of Forces.

14. Define Resolution of Forces.

15. If the magnitude and direction of a force be given show how its effect in another given direction may be determined.

16. Give different methods for determining the resultant of any number of forces acting on a particle.

17. Given the magnitude and direction of the force of the wind upon the sail of a boat, show by means of a diagram the amount of this force which is available in the direction of the boat's motion.

INTRODUCTORY EXAMPLES TO CHAPTER XII.

(*a*) Three forces of 5 lbs., 7 lbs., and 10 lbs., respectively act on a particle in the same line and in the same direction, and another force of 20 lbs. acts on the particle in the opposite direction: what is the resultant force?

By Art. 167, Resultant $= 5 + 7 + 10 - 20 = 2$ lbs. $\therefore$ the resultant is a force of 2 lbs. acting in the direction of the first three forces.

(*b*) Two forces of 10 lbs. and 12 lbs. respectively act on a particle in directions inclined to each other at an angle of 60°: find the resultant.

We may employ the method of Art. 170. (*The Student will draw the diagram*). Let O be the particle. From O draw OB horizontal and $= 12$, and OA at an angle of 60° to OB and $= 10$. Complete the parallelogram and draw the diagonal OL. Then (Art. 169) OL = the resultant. From L draw LD perpendicular to OB produced. Then the triangle LBD is half of an equilateral triangle $\therefore BD = \frac{1}{2}BL = \frac{1}{2}OA = 5$. By Euc. II., 12, $OL^2 = OB^2 + BL^2 + 2OB \cdot BD = 12^2 + 10^2 + 2 \times 12 \times 5 \therefore OL = \sqrt{364} = 19$ lbs. = the resultant.

Or, we may employ the equation of Art. 171,

$R^2 = 10^2 + 12^2 + 2 \times 10 \times 12 \times \cos 60^\circ = 10^2 + 12^2 + 2 \times 10 \times 12 \times \frac{1}{2} = 364 \therefore R = 19$ lbs.

(*c*) The ends of a cord passed through a ring fixed in a wall are pulled each with a force of 10 lbs., the two parts of the cord being inclined to each other at an angle of 120°: find the resultant.

If lines be drawn representing the magnitude and direction of the forces and the parallelogram be completed, it will be found that the parallelogram is a rhombus, whose shorter diagonal representing the resultant is equal to one of the sides. The resultant is therefore 10 lbs.

Or, using the equation of Art. 171,

$R^2 = 10^2 + 10^2 + 2 \times 10 \times 10 \times \cos 120° = 10^2 + 10^2 + 2 \times 10 \times 10 \times (-\frac{1}{2}) = 10^2 + 10^2 - 10^2 = 10^2$ $\therefore R = 10$ lbs.

(*d*) A force of 100 lbs. acts in a direction inclined to the horizontal at an angle of 45°: what is the effect of the force in a horizontal direction?

By Art. 180, the force resolved horizontally $= 100 \cos 45° = 100 \times \frac{1}{2}\sqrt{2} = 50\sqrt{2}$ lbs.

SOLUTIONS OF EXERCISES TO CHAP. XII.

1. Let P and Q be the forces and R the resultant. Then Art. 169 or 171, since their directions form a right angle $R^2 = P^2 + Q^2$. And by question $P : Q :: 3 : 4$ $\therefore 4P = 3Q$. Hence we have the two equations

$P^2 + Q^2 = 225$

$4P = 3Q$

Solving for P and Q, we find $P = 9$ lbs. and $Q = 12$ lbs.

Or:—Since the resultant $= 15$ lbs., the square of the resultant = sum of squares of component forces $= 225$ lbs. And as by the question the components are as $3 : 4$, if we divide 225 into two parts proportional to 3^2 and 4^2, these will be the squares respectively of the components, and extracting the square roots we obtain the forces.

2. If the forces are in the same line then, if they kept the particle at rest, the algebraic sum of the forces would be 0 (Art. 167), that is one of the forces would be equal and opposite to the sum of the other two. But this cannot be the case with the given forces.

If the directions of the forces are not in the same line, then if they kept the particle at rest a triangle could be formed having its sides respectively parallel and proportional to the forces. But (Euc. I., 20), this is impossible with the given forces. Therefore the forces cannot keep the particle in equilibrium.

3. From Art. 174 and Euc. I., 20, the forces cannot be acting in different directions. But (Art. 167), if they are acting in the same line, and the force 12 lbs. in an opposite direction to the forces 8 lbs. and 4 lbs. the particle will remain at rest.

4. (*The Student will draw the diagram.*) Let OAB be the isosceles right-angled triangle whose base AB is the rod, and OA, OB the strings, O being the fixed point. Let T be the tenison in OA, T′ that in OB and W = weight of the rod acting at its middle point since the rod is uniform. Then there are three forces (T, T′ and W) whose directions all pass through the point O; and the sides of the triangle OAB are respectively perpendicular to these forces, OA being at right angles to T′, OB to T, and AB to W. Therefore (Art. 175),

$$T : W :: OB : AB.$$

But since OAB is an isosceles right-angled triangle $OB : AB :: 1 : \surd 2.$

$\therefore T : W :: 1 : \surd 2.$ Similarly it may be shown that $T' : W :: 1 : \surd 2.$

5. (*The Student will draw the diagram.*) Let OAB be the equilateral triangle where AB is the rod and, OA OB the strings, O being the fixed point. Let T be the tension in OA, and T′ that in OB, T being equal evidently to T′. The weight W of AB acts at C, the middle point of AB. Join OC, and from C draw CD parallel to OB and meeting OA in D. Then the sides of the triangle ODC are respectively parallel to the directions of the three forces, OD being parallel to T, CD to T′ and OC to W $\therefore$ (Art. 174)

$$T : W :: OD : OC$$

But if the side of the equilateral triangle be taken $= 1$, then $OD = \frac{1}{2}$, and $OC = \frac{1}{2}\surd 3 \therefore$

$$T : W :: 1 : \surd 3$$

6. By Art. 169 or 170, $R^2 = P^2 + Q^2 \therefore 25^2 = 15^2 + Q^2 \therefore Q = 20$ lbs.

7. Since the square of the resultant is equal to the sum of the squares of the components, it follows from Art. 169 that the angle between the components is a right angle.

Or:—Art. 171,

$R^2 = P^2 + Q^2 + 2PQ \cos a \therefore 35^2 = 21^2 + 28^2 + 2 \times 21 \times 28 \times \cos a \therefore 2 \times 21 \times 28 \times \cos a = 35^2 - (21^2 + 28^2) = 0$ $\therefore \cos a = 0 = \cos 90^\circ \therefore a = 90^\circ$.

8. See Introductory Examples (*c*). The pressure is equal to the resultant of the two tensions. Therefore Art. 171, $R^2 = 20^2 + 20^2 + 2 \times 20 \times 20 \times \cos 120^\circ = 20^2 + 20^2 + 2 \times 20 \times 20 \times (-\tfrac{1}{2}) = 20^2 + 20^2 - 20^2 = 20^2 \therefore R = 20$ lbs.

9. By Art. 180, the effect of the force in a horizontal direction $= 30 \cos 30^\circ = 30 \times \tfrac{1}{2}\sqrt{3} = 15\sqrt{3}$ lbs.

10. By Art. 169 or 170, $R^2 = 12^2 + 4^2 = 160 \therefore R = 4\sqrt{10}$ lbs.

11. Let the directions of the forces be so arranged that the force $12\sqrt{7}$ lbs. will act in a horizontal direction from right to left, the force 20 lbs. being above the horizontal line and the force 30 lbs. below it. Through the particle draw horizontal and vertical axes, and resolve vertically and horizontally as in Art. 181 (*c*). Let X = algebraic sum of forces resolved horizontally and Y = sum resolved vertically. Then,

$X = 20 \cos 60^\circ + 30 \cos 60^\circ - 12\sqrt{7} = 10 + 15 - 12\sqrt{7} = 25 - 12\sqrt{7}$.

$Y = 20 \cos 30^\circ - 30 \cos 30^\circ = -10 \cos 30^\circ = -10 \times \tfrac{1}{2}\sqrt{3} = -5\sqrt{3}$.

Therefore (Art. 181) $R^2 = X^2 + Y^2 = (25 - 12\sqrt{7})^2 + (-5\sqrt{3})^2 = 1708 - 600\sqrt{7} \therefore R = \sqrt{1708 - 600\sqrt{7}} = 11$ lbs. nearly.

Or:—If the directions be arranged so that the force 30 lbs. acts horizontally from left to right, the 20 lbs.

acting above and $12\sqrt{7}$ lbs. below the horizontal, then resolving horizontally and vertically as before, we have

$X = 30 - 20\cos 60^\circ - 12\sqrt{7}\cos 60^\circ = 30 - 20 \times \frac{1}{2} - 12\sqrt{7} \times \frac{1}{2} = 20 - 6\sqrt{7}$.

$Y = 20 \cos 30^\circ - 12\sqrt{7} \cos 30^\circ = 20 \times \frac{1}{2}\sqrt{3} - 12\sqrt{7} \times \frac{1}{2}\sqrt{3} = 10\sqrt{3} - 6\sqrt{21}$.

$R^2 = (20 - 6\sqrt{7})^2 + (10\sqrt{3} - 6\sqrt{21})^2 = 1708 - 600\sqrt{7}$ $\therefore R = \sqrt{121}$ nearly $= 11$ lbs. nearly.

12. By Art. 171, $R^2 = 10^2 + 12^2 + 2 \times 10 \times 12 \times \frac{1}{6}$ $\therefore R = 16 \cdot 9$ lbs.

13. By Art. 171, $R^2 = 2^2 + 3^2 + 2 \times 2 \times 3 \times \cos 45^\circ = 4 + 9 + 12 \times \frac{1}{2}\sqrt{2}$ $\therefore R = \sqrt{13 + 6\sqrt{2}} = 4 \cdot 6$ lbs. nearly.

14. By Art. 171, $R^2 = P^2 + Q^2 + 2PQ \cos 135^\circ$, and by question $R = Q$ $\therefore Q^2 = P^2 + Q^2 + 2PQ \times -\frac{1}{2}\sqrt{2}$ $\therefore 0 = P^2 - PQ\sqrt{2}$ $\therefore P^2 = PQ\sqrt{2}$ $\therefore \frac{P}{Q} = \frac{\sqrt{2}}{1}$ $\therefore P : Q :: \sqrt{2} : 1$.

15. See Fig. of Example 4, page 125, Text-Book. The angle APB is in this case 45°. By Art. 174, the forces are proportional to the sides of the triangle PAB, to which they are respectively parallel. Therefore $F : 20 :: AB : AP$. But PAB is an isosceles right-angled triangle $\therefore AB = AP$ $\therefore F = 20$ lbs.

16. The pull on the point P is the force acting along the strings BP; and as the three forces acting on the point B are in equilibrium, it follows that the tension in BP must be equal and opposite to the resultant of the two forces F and the 20 lbs. weight. But by Art. 171 this resultant $= \sqrt{20^2 + 20^2} = 20\sqrt{2}$ lbs., which is the pull on the fixed point.

17. Art. 171, $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$, therefore

$(5\sqrt{3})^2 = 5^2 + 5^2 + 2 \times 5 \times 5 \times \cos \alpha$ $\therefore$

$50 \cos \alpha = 75 - (25 + 25) = 25$ $\therefore \cos \alpha = \frac{1}{2} = \cos 60^\circ$ $\therefore \alpha = 60^\circ$.

18. Each peg has to sustain two tensions of 10 lbs. each in directions inclined at an angle of 60°. The resultant pressure on each peg being R, we have, by Art. 171,

$$R^2 = 10^2 + 10^2 + 2 \times 10 \times 10 \times \tfrac{1}{2} \therefore R = \sqrt{10^2 \times 3}$$
$$= 10\sqrt{3} \text{ lbs.}$$

19. See Solution to Exercise 2.

20. Determining the resultant by the method of Art. 177, it is evident that the resultant is represented by the line joining the extremities of the first and third sides, and this line is one of the diagonals of the hexagon.

21. See the Fig. to Example 6, p. 125, Text-Book. In this case $AC = 6$ feet, $BC = 8$ feet, and $AB = 10$ feet. Let t = tension in AC and t' that in BC. Then (Art. 175), the forces are proportional to the sides to which they are respectively pependicular $\therefore$

$$t : 20 \text{ lbs.} :: 8 : 10 \therefore t = 16 \text{ lbs.}$$

$$t' : 20 \text{ lbs.} :: 6 : 10 \therefore t' = 12 \text{ lbs.}$$

22. By Art. 171, $R^2 = 5^2 + 4^2 + 2 \times 5 \times 4 \times \frac{1}{10} \therefore R = \sqrt{45}$ $= 3\sqrt{5}$ lbs.

23. By Art. 171, $R^2 = 10^2 + 12^2 + 2 \times 10 \times 12 \times \cos 120^\circ \therefore$ $R = \sqrt{100 + 144 + 2 \times 10 \times 12 \times -\frac{1}{2}} = \sqrt{124}$ ozs.

24. By Art. 171, $R^2 = 12^2 + 16^2 + 2 \times 12 \times 16 \times \cos 90^\circ$. And since $\cos 90^\circ = 0 \therefore R = \sqrt{12^2 + 16^2} = 20$ ozs.

25. Let the forces be arranged so that the 4 lbs. force will act horizontally towards the right, the 6 lbs. above the horizontal line, its direction making an angle of 60° with the former, and the 8 lbs. also above the horizontal making an angle of 90° with the second. Then resolving horizontally and vertically as in Art. 181, and in Solution to Exercise 11, we have

$$X = 4 + 6 \cos 60^\circ - 8 \cos 30^\circ = 4 + 3 - 4\sqrt{3} = 7 - 4\sqrt{3}$$
$$Y = 6 \cos 30^\circ + 8 \cos 60^\circ = 3\sqrt{3} + 4.$$
$$\therefore R^2 = (7 - 4\sqrt{3})^2 + (3\sqrt{3} + 4)^2 \therefore R = \sqrt{140 - 32\sqrt{3}} =$$
$$9 \cdot 2 \text{ lbs. nearly.}$$

26. This follows as a corollary from Art. 175 (2).

27. The resultant may be found by any of the methods described in Art. 181.

28. See Art. 180. Resolving horizontally we obtain 100 cos $30^\circ = 50\sqrt{3}$ lbs. Resolving vertically, the effect in a vertical direction $= 100 \cos 60^\circ = 50$ lbs.

29. See Solutions to Exercises 11 and 25. Let the forces be so arranged that the 100 lbs. force acts horizontally towards the right, the 99 lbs. force above, and the 101 lbs. below the horizontal line. Then resolving horizontally and vertically as in Art. 181, we have

$$X = 100 - 99 \cos 60^\circ - 101 \cos 60^\circ = 100 - 99 \times \tfrac{1}{2} - 101 \times \tfrac{1}{2} = 0.$$

$$Y = 99 \cos 30^\circ - 101 \cos 30^\circ = 99 \times \tfrac{1}{2}\sqrt{3} - 101 \times \tfrac{1}{2}\sqrt{3} = -\sqrt{3}.$$

The resolved forces in a horizontal direction vanish, therefore the resultant is the sum of the resolved parts in the vertical direction. The resultant therefore $= \sqrt{3}$ lbs. acting at right angles to the direction of the force 100 lbs. The − sign attached to the resultant merely indicates that in the particular arrangement chosen the direction of the resultant lies *below* the horizontal line.

Proceeding in the same way with any arrangement of the directions of the forces the equations obtained in resolving in two directions will differ from those above, but the same results will be arrived at.

30. By Art. 171, $R^2 = 2^2 + (3\sqrt{3})^2 + 2 \times 3\sqrt{3} \times 2 \times \tfrac{1}{2}\sqrt{3} = 49 \therefore R = 7$ lbs,

31. By Art. 171, since $\cos 135^\circ = -\tfrac{1}{2}\sqrt{2}$, we have

$$R^2 = 17^2 + (24\sqrt{2})^2 - 2 \times 17 \times 24\sqrt{2} \times \tfrac{1}{2}\sqrt{2} \therefore R = 25 \text{ lbs.}$$

32. Art. 171, $R^2 = P^2 + Q^2 + 2PQ \cos \alpha \therefore 14^2 = P^2 + 4^2 + 2 \times P \times 4 \times \cos 30^\circ \therefore P^2 + 4\sqrt{3}P = 180 \therefore P = -2\sqrt{3} \pm\sqrt{(2\sqrt{3})^2 + 180} = -2\sqrt{3} \pm 8\sqrt{3} = 6\sqrt{3} = 10{\cdot}39$ lbs.

33. Since the pegs are smooth the tension throughout the string is the same and is equal to 10 lbs. The pressure on each peg is the resultant of the two tensions, each of which is 10 lbs. These tensions at the peg A are inclined to each other at an angle of 60°, and at each of the other pegs at an angle of $(90^\circ + 60^\circ) = 150^\circ$. Therefore the resultant pressure at A being called R, we have, by Art. 171,

$R^2 = 10^2 + 10^2 + 2 \times 10 \times 10 \times \cos 60^\circ \therefore R = \sqrt{300} = 10\sqrt{3}$ lbs.

And the resultant pressure on each of the pegs B and C being called R′ we have, by Art. 171,

$R'^2 = 10^2 + 10^2 + 2 \times 10 \times 10 \times \cos 150^\circ = 100 + 100 + 200\left(-\frac{\sqrt{3}}{2}\right) = 100(2 - \sqrt{3}) \therefore R = 10\sqrt{2 - \sqrt{3}}$ lbs.

34. The angle at A being 90°, the angles ABC and ACB are each 45°, and therefore the tensions at B and those at C are in each case inclined at angles of $(90^\circ + 45^\circ) = 135^\circ$. Hence proceeding as in last solution, we have

$R^2 = 10^2 + 10^2 + 2 \times 10 \times 10 \times \cos 90^\circ = 100 + 100 = 200 \therefore R = 10\sqrt{2}$ lbs., the pressure on A.

$R'^2 = 10^2 + 10^2 + 2 \times 10 \times 10 \times \cos 135^\circ = 100 + 100 + 200(-\frac{1}{2}\sqrt{2}) = 100(2 - \sqrt{2}) \therefore R = 10\sqrt{2 - \sqrt{2}}.$

35. See Introductory Examples (*c*). The pressure on the peg is the resultant of the tensions, therefore by Art. 171,

$R^2 = 80^2 + 80^2 + 2 \times 80 \times 80 \times \cos 60^\circ = 80^2 + 80^2 + 2 \times 80 \times 80 \times \frac{1}{2} = 3 \times 80^2 \therefore R = 80\sqrt{3}$ lbs.

36. Compounding, by Art. 167, each pair of forces acting in opposite directions we find,

Resultant of 3 lbs. and 5 lbs. forces = 2 lbs. acting in the direction of the 5 lbs. force.

Resultant of 4 lbs. and 6 lbs. forces = 2 lbs. acting in the direction of the 6 lbs. force.

The forces are therefore equivalent to two forces each 2 lbs., and acting in directions inclined at a right angle. Therefore, Art. 170,

$R^2 = 2^2 + 2^2 \therefore R = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ lbs.

37. (*The Student will draw the diagram.*) Let the tension in OB be called t_1 and that in OA, t_2. There are three forces acting at O, the two tensions and the weight 40 lbs. Each of these forces is at right angles to a side of the triangle AOB, therefore by Art. 175

$t_1 : 40 :: 12 : 20 \therefore t_1 = 24$ lbs.

$t_2 : 40 :: 16 : 20 \therefore t_2 = 32$ lbs.

38. From one extremity of one of the diagonals of the given parallelogram, draw a line equal and parallel to the other diagonal, and complete the new parallelogram. The diagonal of this parallelogram, drawn from the extremity of the first diagonal, will represent the resultant of the two given forces, and from the construction it will be at once seen that this diagonal is double one of the sides of the given parallelogram.

39. The directions of the forces form three adjacent edges of a parallelopiped $\therefore$ by Art. 178, $R = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{77}$ lbs.

40. In solving this problem the forces may be resolved in any two directions, the equations obtained varying with the directions. The following arrangement may be adopted. The student will draw the diagram.

Take a point O and draw OA horizontal and equal to 10 lbs., OB above the horizontal line making an angle of 60° with OA and equal to 10 lbs., OC above the horizontal at an angle of 75° with OB and equal to 8 lbs., OD at an angle of 60° with OC and equal to 6 lbs., and OE equal to $4\sqrt{3}$ lbs. and at an angle of 90° with OD. Through O draw vertical and horizontal axes. Then it will be readily seen that OB and OC make angles of 45° and 30° respectively with the vertical, that OD is horizontal and that OE is vertical and below the horizontal.

Resolving horizontally and vertically, and calling the sum of the resolved parts in a horizontal direction X, and the sum of those in a vertical Y, we have

$$X = 10 + 10\cos 45^\circ - 8\cos 60^\circ - 6 = 10 + 5\sqrt{2} - 4 - 6 = 5\sqrt{2}$$

$$Y = 5\sqrt{2} + 4\sqrt{3} - 4\sqrt{3} = 5\sqrt{2}$$

Therefore, Art. 181, $R = \sqrt{X^2 + Y^2} = \sqrt{(5\sqrt{2})^2 + (5\sqrt{2})^2} = 10$ lbs.

41. Resolving each force horizontally by Art. 180, and adding, we have

$$\text{Force} = 30\cos 60^\circ + 20\cos 45^\circ = 30 \times \tfrac{1}{2} + 20 \times \tfrac{1}{2}\sqrt{2} = 15 + 10\sqrt{2} = 29\cdot 14 \text{ lbs.}$$

42. The pull on the rope when resolved in the direction of the boat's motion is the effective force acting on the boat in the direction it is moving. This resolved force is equal to the force of the pull multiplied by the cosine of the angle made by the direction of the rope and the direction of the motion of the boat. With a long rope the angle between the direction of the pull and the line of motion of the boat can be smaller than with a shorter rope. The smaller the angle the greater its cosine, and therefore with the same pull the effective force acting in the direction of the motion of the boat is greater with a long rope than with a short one.

43. (*The Student will draw the diagram.*) Let O be the weight = 24 lbs. Draw OB to represent the horizontal string, and draw OA on the other side of the vertical through O and making an angle of 30° with the vertical to represent the second string. Let t_1 be the tension in OB and t_2 the tension in OA. Through A draw AC horizontal and meeting the vertical in the point C. Then the sides of the triangle ACO are respectively parallel to the three forces acting on O, AC being parallel to t_1, AO to t_2, and CO to the weight 24 lbs. And since the triangle ACO is evidently half an equilateral triangle, if the side AO be taken = 1, AC = $\frac{1}{2}$, and CO = $\frac{1}{2}\sqrt{3}$. Therefore, by Art, 174,

$$t_1 : 24 :: AC : CO :: 1 : \sqrt{3} \therefore t_1 = \frac{24}{\sqrt{3}} = 8\sqrt{3} \text{ lbs.}$$

$$t_2 : t_1 :: AO : AC :: 1 : \tfrac{1}{2} \therefore t_2 = 2t_1 = 16\sqrt{3} \text{ lbs.}$$

CHAPTER XIII.

EXAMINATION QUESTIONS.

1. What are Parallel Forces? How are they divided?

2. What is meant by the *transmissibility of force?*

3. Two given parallel forces act on a body: show how the magnitude, direction, and point of application of their resultant may be determined (*a*) when the forces are like, (*b*) when they are unlike.

4. How may the preceding results be experimentally verified?

5. Define the term Couple. Show that a couple has no single resultant.

6. How may the magnitude, direction, and point of application of the resultant of any number of parallel forces be determined?

7. What is the centre of a system of parallel forces?

8. In what case has a system of parallel forces no single resultant? In what case does the resultant vanish?

9. Show how a given force may be resolved into two or more component parallel forces.

INTRODUCTORY EXAMPLES TO CHAP. XIII.

(*a*) Like parallel forces of 12 lbs. and 3 lbs. act respectively at the extremities of a rod $2\frac{1}{2}$ feet long: find the resultant and its point of application.

> By Art. 185, Resultant $= 12 + 3 = 15$ lbs. To find its point of application we divide the distance between the forces, 30 inches, inversely as the forces. Therefore, by the rule of proportional parts, as $12 + 3 : 12 :: 30 : 24$ inches, the distance from 3 lbs. force. And $30 - 24 = 6$ inches, the distance from the 12 lbs. force.

(*b*) If the forces in the last question were unlike, what would be the magnitude and direction of the resultant?

By Art. 186, Resultant $= 12 - 3 = 9$ lbs., acting in the direction of the 12 lbs. force. To find the point of application of the resultant we produce the line between the forces through the point of application of the greater force, so that the whole line produced is to the produced part as the greater force is to the smaller. Let the produced part be called x, then by Art. 186,

As $(30 + x) : x :: 12 : 3 \therefore 12x = 90 + 3x \therefore x = 10$ inches, the distance of the point of application of the resultant from the greater force.

(*c*) The resultant of two like parallel forces is 20 lbs.; one of the forces is 15 lbs., and it acts at a distance of 4 inches from the resultant: what is the other force, and where does it act?

By Art 185, the other component $= 20 - 15 = 5$ lbs. And since the distances of the components from the resultant are inversely proportional to the forces, we have

As 5 lbs. : 15 lbs. :: 4 inches : 12 inches, the distance of the 5 lbs. force from the resultant.

SOLUTIONS OF EXERCISES TO CHAP. XIII. PAGE 134.

1. By Art. 185, the resultant $= 10 + 6 = 16$ lbs. To find the point of application we divide the distance between the forces inversely as the forces. Hence, using the rule of proportional parts, as $10 + 6 : 10 :: 4 : 2\frac{1}{2}$ feet, the distance from the 6 lbs. force, and $4 - 2\frac{1}{2} = 1\frac{1}{2}$ feet, the distance from the 10 lbs. force.

2. See last solution. The pressure upon the point will be the resultant of the forces acting on the rod, and the point of application of the resultant will be the point on which the

rod will balance, since then the reaction of the fixed point will be equal and opposite to the resultant of the forces, and hence the system will be in equilibrium.

By Art. 185, the resultant $= 5 + 7 = 12$ lbs. = pressure on fixed point. The position of this point may be found as in last exercise, or as follows:—Let x = distance of point of application of resultant from the force of 5 lbs., then the distance of this point from the other force $= 30 - x$ $\therefore$ by Art. 185,

$5 : 7 :: 30 - x : x \therefore 5x = 210 - 7x.$

$\therefore 12x = 210 \therefore x = 17\frac{1}{2}$ inches from the force of 5 lbs. And $30 - 17\frac{1}{2} = 12\frac{1}{2}$ inches, the distance from the 7 lbs. force.

3. By Art. 186, the resultant $= 16 - 12 = 4$ lbs. Let x = the distance beyond the point of application of the greater force at which the resultant acts, then, by Art. 186,

$1 + x : x :: 16 : 12 \therefore 16x = 12 + 12x \therefore x = 3$ feet. Hence the resultant acts at a point 3 feet distant from the point of application of the greater force.

4. See solution to last exercise. Resultant $= 10 - 7 = 3$ lbs. Let x = distance to which the line between the forces is produced through the direction of the greater force, then, by Art. 186,

$6 + x : x :: 10 : 7 \therefore x = 14$ inches from greater force.

5. (*The Student will form the diagram.*) Draw a square ABCD, having at its angular points A, B, C, D like parallel forces of 1, 2, 3, and 6 lbs. respectively acting. Draw the diagonals AC and BD intersecting in O.

By Art. 189, the resultant of all the forces $= 1 + 2 + 3 + 6 = 12$ lbs.

By Art. 185, the resultant of the forces 1 lb. and 3 lbs. = 4 lbs. acting at a point E on AC at one fourth the length of AC from C. Similarly the resultant of the forces at B and D is 8 lbs. acting at a point F distant one fourth of BD from the point D. Join FE and divide it at G inversely as the forces 8 lbs. and 4 lbs. $\therefore$ G is the point of application of the resultant of all the forces (Art. 189). Now since the line FE bisects the sides of the triangle ODC, it bisects the perpendicular from O on the line DC, and it is therefore $2\frac{1}{2}$ feet distant from DC. The point G may also be easily shown to be $4\frac{1}{6}$ feet distant from AD.

NOTE.—A better method of solving problems similar to the above is explained in Chap. XV.

6. We resolve the force 224 lbs. into two components acting at distances which are in the ratio of 2 to 1. By Art. 185, the components are inversely as these numbers. Let $x =$ the less component, then $224 - x =$ the greater.

And, Art. 185, $x : 224 - x :: 1 : 2 \therefore 2x = 224 - x \therefore x = 74\frac{2}{3}$ lbs., the less component; and $224 - 74\frac{2}{3} = 149\frac{1}{3}$ lbs., the greater.

7. (*The Student will draw the diagram.*) Let each of the equal forces be P, acting respectively in the directions AB, BC, and CA. By the principle of Art. 184 the force acting along BC may be made to act at C and in the direction of BC produced, and similarly the force acting along CA may act from the point C. At C therefore there are two forces each $=$ P acting in directions at right angles to each other. Compounding these forces (Art. 169) their resultant is the diagonal of a square whose side $=$ P. The resultant therefore $= \sqrt{2P^2} = P\sqrt{2}$; and by the construction it is seen that this resultant is parallel to AB. The three forces are consequently equivalent to two—a force $= P\sqrt{2}$ acting at C in a parallel direction to AB, and a force P acting along AB and which may be made to act at D. There are therefore two unlike parallel forces, $P\sqrt{2}$ and P, acting at the points C and D respectively, and by Art. 186 their resultant acts at a point found by producing DC to a point E such that DE, the whole line produced, is to CE, the produced part, $:: P\sqrt{2} : P :: \sqrt{2} : 1$.

8. (*The Student will draw a diagram similar to that of Example 2, page 134, Text-Book.*) Let the three forces be P, Q, and S acting at the points A, B, and C respectively. Then, as in Example 2, page 134 of Text-Book, it can be shown that D is the point of application of the resultant of P and Q. At D therefore we have a force P + Q proportional to the sum of the triangles BOC, AOC, and at C we have a force S proportional to the sum of triangles BOD, AOD ∴

P + Q : S : : BOC + AOC : BOD + AOD. But by Euc. VI., 1, and the principles of proportion it may be readily shown that

BOC + AOC : BOD + AOD : : CO : OD ∴

P + Q : S : : CO : OD, and therefore by Art. 185, O is the point of application of the resultant of the forces P + Q and S, and is therefore the point of application of the resultant of the three given forces.

9. Since one of the components is 3 inches distant from the resultant 8 lbs., the other component is, by the question, 5 inches distant. Hence dividing 8 lbs. into parts proportional to 5 and 3 we find 5 lbs. and 3 lbs. the components.

10. Let x = smaller force, then, Art. 186, $x + 3$ = greater. And by Art. 186, as

$6 + 8 : 8 :: x + 3 : x \therefore 14x = 8x + 24 \therefore x = 4$ lbs., and $4 + 3 = 7$ lbs., the greater force.

CHAPTER XIV.

EXAMINATION QUESTIONS.

1. Explain what is meant by the moment of a force.

2. How is a moment measured (*a*) geometrically; (*b*), algebraically?

3. How are moments divided?

4. If the directions of two forces meet at a point, their moments with respect to any point situated on the line of action of the resultant are equal and unlike. Give the proof.

5. State the converse of the foregoing proposition, and give its proof.

6. If the directions of two forces meet in a point, the algebraic sum of the moments of the forces round any point in their plane is equal to the moment of their resultant round the same point. Give the proof: (*a*) when the forces are like; (*b*), when unlike.

7. Show that the proposition stated in question 4 follows as a corollary from that in question 6.

8. Show that if any number of forces in the same plane act on a particle, the algebraic sum of their moments round any point in the plane is equal to the moment of the resultant round the same point.

9. Show that if any number of forces acting on a particle be in equilibrium, the algebraic sum of their moments with respect to any point in the plane is equal to zero.

10. The moments of two like parallel forces round any point in the direction of their resultant are equal and unlike. Give the proof.

11. State and prove the converse of the foregoing proposition.

12. The algebraic sum of the moments of any two parallel forces round any point in their plane is equal to the moment of their resultant round the same point. Give the proof.

13. Show that the proposition stated in question 10 follows as a corollary from that in question 12.

14. Show that the algebraic sum of the moments of any number of parallel forces round any point in the plane is equal to the moment of the resultant round the same point.

15. Show that if any number of parallel forces be in equilibrium the algebraic sum of their moments round any point in their plane is equal to zero.

16. Define the following terms: *couple*, *like* and *unlike couples*, *arm of couple*, *moment of couple*, *axis of couple*.

17. The algebraic sum of the moments of the two forces which form a couple is constant round every point in their plane. Give the proof.

18. Show that two unlike couples in the same plane whose moments are equal will balance each other.

19. State some inferences from the last proposition regarding the effects of couples in the same plane.

20. Show that if a system of forces in one plane act on a particle they are equivalent either to one single resultant force or they are in equilibrium.

21. If a system of forces in one plane acting on a particle be in equilibrium, show that (1), the sum of the resolved parts of the forces in any direction vanishes; (2), the sum of the moments of the forces round any point in the plane vanishes.

22. If three forces are in equilibrium, and are not parallel show that their directions all pass through the same point.

23. Show that a system of forces in one plane (1), are equivalent to a single resultant; or (2), are in equilibrium; or (3), are equivalent to a couple.

24. If a system of forces in one plane are in equilibrium, show that (1), the algebraic sum of the forces resolved in any direction vanishes; (2), the algebraic sum of the moments of the forces round any point in the plane vanishes.

25. State the three conditions which determine the equilibrium of a system of forces in one plane. Show that if any one of these three conditions hold with regard to a system of forces in one plane, these forces are in equilibrium.

26. In dealing with problems containing unknown forces, how may we eliminate any one that we do not require in the investigation?

INTRODUCTORY EXAMPLES TO CHAP. XIV.

(*a*) Two unlike parallel forces of 10 lbs. and 12 lbs. act on a body: what are the measures of the moments respectively of these forces round a point in the body situated between the directions of the forces, 4 inches distant from the direction of the first force and 6 inches from that of the second?

See Arts. 195, 196, and 197.

The moment of the force of 10 lbs. is measured by the product $10 \times 4 = 40$, and that of the 12 lbs. by $12 \times 6 = 72$. But since the moments are unlike, if the first be called positive, the second will be negative. The measures of the moments, therefore, are 40 and -72.

The meaning to be attached to these measures is, that the effect of the first force in producing rotation round the fixed point bears to the effect of the second force in producing rotation round the same point the ratio that the number 40 bears to the number 72. The $-$ sign attached to the latter measure merely denotes that the rotation produced by the second force is opposite in direction to that produced by the first.

(*b*) A bar supposed to be without weight is 4 feet long, and three like parallel forces act on it; a force of 1 lb. at one end, 2 lbs. at a distance of a foot from that end, and 3 lbs. at the other end: upon what point will the bar balance, and what will be the pressure upon the fixed point?

The resultant of the forces passes through a point in the bar which if supported the system will be in equilibrium, since the reaction of the fixed point will then be equal and opposite to the resultant of the forces. By Art. 189, the resultant $= 1 + 2 + 3 = 6$ lbs. This, therefore, is the pressure upon the fixed point. That point or fulcrum must evidently be situated between the points of application of the forces 2 lbs. and 3 lbs. Let $x =$ its distance from the 3 lbs. force, then its distance from the 1 lb. force $=$ $(x - 4)$ feet, and from the 2 lbs. $= (x - 3)$ feet. If the forces act at right angles to the bar these distances are the respective perpendiculars from the fulcrum upon the directions of the forces, and whatever be the directions of the forces these distances are proportional to the perpendiculars. Taking moments round the fulcrum, since the moment of the resultant round this point is 0, the algebraic sum of the moments of the forces round the fulcrum must be 0 (Art. 204), and therefore the moment of the 3 lbs. must be equal and opposite to the sum of the moments of the other two. Therefore

$3x = 1(4 - x) + 2(3 - x) \therefore x = 1\frac{2}{3}$ feet from the end where the 3 lbs. force acts.

NOTE.—The same result will be obtained by taking moments round any point, the distances of which from the forces are known, and applying the proposition of Art. 204.

SOLUTIONS OF EXERCISES TO CHAP. XIV. PAGE 146.

1. See Art. 174. Compounding two of the forces (Art. 169) their resultant is in the same plane as the two components. But since the three forces are in equilibrium this resultant must be equal and opposite to the third force; therefore the third force is in the same plane as the other two.

2. See Art. 208. When the couples are unlike and have equal moments they produce equal and opposite effects.

3. If they are not parallel forces, the directions of at least two of the forces must meet at a point. At this point compound the two forces and find the resultant. This resultant is equal and opposite to the third force, and therefore the direction of the latter passes through the point of intersection of the directions of the two former.

> If, therefore, the directions of any two of the forces are not parallel, the directions of all the forces must pass through the same point. If the directions do not pass through the same point, and if the forces are in equilibrium, then they must be all parallel; two of them must act in the same direction, and the third —equal in magnitude to their sum—must act in the opposite direction, at a point found by dividing the distance between them inversely as the forces.

4. (*The Student will draw the diagram showing the position of the forces.*) If a sixth force acted at the unoccupied angle, the resultant of all the forces would act at the centre of the hexagon. Removing the sixth force, it is evident from the symmetrical arrangement that the C.G. of the five forces will be some point on the diagonal passing through the unoccupied angle, and on the other side of the centre. Let $x =$ its distance from the centre, let each force be called P, and take moments round the unoccupied angle (Art. 204). If the forces be taken acting at right angles to the diagonal passing through the unoccupied angle, it will be readily seen that the direction of one of the forces is at a perpendicular distance of 20 inches from the point round which moments are taken, those of two are at a perpendicular distance of 15 inches, and those of two at a perpendicular distance of 5 inches. And whatever be the direction of the parallel forces, these distances will be proportional to the perpendiculars respectively from the point upon the directions of the forces. By Art. 204 the moment of the resultant round the unoccupied angle is equal to the sum of the moments of the forces round the same point $\therefore$

> $5Px = P \times 20 + 2P \times 15 + 2P \times 5 \therefore 5x = 60 \therefore x = 12$ inches from the unoccupied angle.

NOTE.—A better and more general method of finding the C.G. is explained and illustrated in Chap. XV. By this method the position of the C.G. is determined by finding its distance from each of two fixed lines in the plane.

5. If the forces act in a direction at right angles to the bar, the distances given in the question will be respectively the lengths of the perpendiculars from the end upon the directions of the forces; and whatever be the direction of the parallel forces, the perpendiculars upon their directions will be respectively proportional to the given distances.

Let $x =$ the distance of the resultant from the given end of the bar; then, Art. 189, the resultant $= 1 + 2 + 4 + 7 = 14$ lbs. Taking moments round the end, we have, by Art. 204,

$14x = 1 \times 3 + 2 \times 5 + 4 \times 7 + 7 \times 9 \therefore x = 7\frac{3}{7}$ feet from end.

6. Proceeding exactly as in last solution, we have resultant $= (3 + 4 + 5 + 7)$ lbs. $= 19$ lbs. And by Art. 204,

$19x = 3 \times 2 + 4 \times 4 + 5 \times 6 + 7 \times 8 = 108 \therefore x = 5\frac{13}{19}$ feet from end of bar.

7. Proceeding as in solution to Exercise 5, we have resultant $= 2 + 6 + 4 = 12$ lbs. Taking moments round the first end $\therefore$ (Art. 204),

$12x = 2 \times 0 + 6 \times 6 + 4 \times 12 = 84 \therefore x = 7$ inches from end.

8. Let a point be taken *within* the polygon and let the moments round this point of the forces represented by the sides of the polygon be determined geometrically. Then the moments are all like, and each moment is twice the area of the triangle having the point for vertex and the line representing the force for base; therefore the sum of all the moments is equal to twice the area of the polygon. Again let a point be taken *without* the polygon and the moments determined as before. These moments are not all like, but their algebraic sum is readily seen to be once more twice the area of the polygon. Therefore the algebraic sum of the moments of the forces round *any* point in the plane is constant, and therefore (Art. 207) the forces are equivalent to a couple.

9. (*The Student will draw a diagram of the arrangement.*) Let the reaction at A be R acting vertically upwards, and that at B, R′ acting at right angles to CB. From A draw a perpendicular upon the direction of R′, and from the middle point of the beam draw a line vertically downwards representing the direction of the weight W. Then it may be easily shown that the angles CAB, CBA are each 30°; that the perpendicular from A upon the direction of R′ makes an angle of 30° with the beam, and therefore the perpendicular $= l \cos 30^\circ$; and that the direction of the weight W cuts the line AC perpendicularly, and therefore the perpendicular from A on the direction of $W = \frac{1}{2} l \cos 30^\circ$. Then resolving vertically and horizontally and taking moments round A (see Art. 214 and Exam. 3, page 145 Text-Book),

$$R + R' \cos 60^\circ - W = 0 \quad \dots\dots\dots \quad (1)$$

$$R' \cos 30^\circ - T = 0 \quad \dots\dots\dots\dots \quad (2)$$

$$R' \, l \cos 30^\circ - W \times \tfrac{1}{2} l \cos 30^\circ = 0 \quad \dots \quad (3)$$

From equation (3) $R' = \frac{1}{2}W$. Substituting this value in equation (2), we find $T = \frac{1}{2}W \cos 30^\circ = \frac{1}{2}W \times \frac{1}{2}\sqrt{3} = \dfrac{W\sqrt{3}}{4}$

NOTE.—Resolving vertically the horizontal force vanishes, resolving horizontally the vertical forces vanish, and taking moments round A, the forces acting through this point vanish.

10. As the distance between the man's hand and shoulder diminishes the distance between the shoulder and the bundle increases, the greater consequently becomes the moment of the bundle, and the greater must be the pull at the other end in order to maintain equilibrium. The pressure on the shoulder is (Art. 185) the sum of the forces at the extremities of the stick, and consequently this pressure increases with the increase of the moment of the bundle. This pressure on the shoulder is met and equilibrated by an equal opposite reaction of the shoulder, so that the pressure on the ground, which is always the weight of the man, bundle, and stick, is not affected by any change in the position of the bundle.

> If the man placed himself in one of the scale pans of a weighing crane, and was equilibrated with weights in the other scale, and if he pressed downwards with his hand on his shoulder or other part of his body, this pressure would be counteracted by an equal and opposite reaction, but his weight or pressure on the scale pan would not be affected.

11. Since the rod is uniform its weight acts at its middle point. Let the weight be w. The pressure on the peg at the extremity is met by an equal reaction acting downwards, and the other peg bears a pressure equal to the sum of this reaction and w. The weight w acts at a point 2 inches from the latter peg, and the reaction at a point 4 inches distant $\therefore$ (Art. 185) the reaction at the extremity $= \frac{1}{2}w$. The pressure on the peg near the middle of the rod is therefore (Art. 185) equal to $w + \frac{1}{2}w = 1\frac{1}{2}w$; and the ratio of the pressures on the pegs is as $1\frac{1}{2}w : \frac{1}{2}w$, or as $3 : 1$.

12. Let A and B be the two pegs. Let x = distance from the peg B where the weight 24 ozs. is attached, thereby making the pressure on A only 8 ozs. At A there is a reaction upwards of 8 ozs., the moment of which round B *plus* the moment of the 24 ozs. round B (both of these moments being like) is balanced by the moment round the same point of the 20 ozs. acting at the middle point. Therefore, taking moments round B,

> $24x + 8 \times 12 = 20 \times 6 \therefore x = 1$ inch from B or 5 inches from the end of the rod.

13. See Solution to Exercise 9. The diagram being drawn it will be seen that there are four forces, the weight of the beam, the tension of the string, and the reactions of the smooth planes. Let R be the reaction at A, and R′ that at B. Then it is easily shown that R′ makes an angle of 60° with the vertical and 30° with the horizontal through the point B; that the beam makes angles of 30° with the planes AC, CB; and that the perpendicular from A on the direction

of R′ makes an angle of 30° with the beam AB. Resolving vertically and horizontally and taking moments round A (Arts. 214 and 216), we have

$R + R' \cos 60° - 100 = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. (1)$

$T - R' \cos 30° = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. (2)$

$R' \times AB \cos 30° - W \times \frac{1}{2}AB \cos 30° = 0 \;.\;.\;.\;. (3)$

From equation (3), $R' = \frac{1}{2} \times 100 = 50$ lbs.

From (2), $T = R' \cos 30° = 50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}$ lbs.

From (1), $R = 100 - R' \cos 60° = 100 - 50 \times \frac{1}{2} = 75$ lbs.

14. See Solution to last exercise. Draw diagram as in last solution. Resolve vertically and horizontally and take moments round A. Then, Art. 214,

$R + R' \cos 60° - 32 = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. (1)$

$T - R' \cos 30° = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. (2)$

$R' \times AB \cos 30° - 32 \times \frac{1}{2}AB \cos 30° = 0 \;.\;.\;. (3)$

From (3), $R' = 32 \times \frac{1}{2} = 16$ lbs. From (2), $T = 16 \times \frac{\sqrt{3}}{2} = 8\sqrt{3}$ lbs.

From (1), $R = 32 - 16 \times \cos 60° = 32 - 16 \times \frac{1}{2} = 24$ lbs.

15. Draw AC horizontal and = 3, CB below the horizontal inclined at an angle of 120° to AC and = 12. At A let the weight of 20 ozs. be suspended, and let x be the weight attached to B so that there may be equilibrium with this position of the bent lever. From B draw the perpendicular BD on AC produced through C. Then since the angle ACB = 120°, DCB = 60°, and therefore CD = $\frac{1}{2}$CB = 6 inches. Since there is equilibrium, the resultant passes through the fulcrum C $\therefore$ (Art. 198) the moments of the two weights round C are equal and opposite $\therefore$

$x \times 6 = 20 \times 3 \;\therefore\; x = 10$ ozs.

16. In this arrangement in order that the beam may remain in the position stated, and that the cord BC may

keep vertical, the beam AB may be imagined to rest with its end B against a smooth vertical wall directly under the point C in the wall to which the cords are attached. There are therefore four forces, the two tensions, the weight of the beam, and the reaction at B, which is horizontal. Let R be this reaction, let the tension in BC be T, and that in AC be T′. Then the angles CAB, ACB are each 45°, and the perpendicular from B on $AC = AB \cos 45°$. Resolving vertically and horizontally and taking moments round B, we have

$$T + T' \cos 45° - 50 = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$T' \cos 45° - R = 0 \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$T' \times AB \cos 45° = 50 \times \tfrac{1}{2}AB \quad \ldots\ldots\ldots\ldots \quad (3)$$

From (3) $T' \times \frac{1}{2}\sqrt{2} = 50 \times \frac{1}{2} \therefore T' = 25\sqrt{2}$ lbs.

From (1) $T = 50 - 25\sqrt{2} \times \frac{1}{2}\sqrt{2} = 25$ lbs.

Or :—Taking moments round B and round A, we have

Round B, $50 \times \frac{1}{2}AB - T' \times AB \cos 45° = 0 \therefore T' \cos 45°$

$$= 50 \times \tfrac{1}{2} \therefore T' = \frac{50}{\sqrt{2}} = 25\sqrt{2} \text{ lbs.}$$

Round A, $50 \times \frac{1}{2}AB - T \times AB = 0 \therefore T = 50 \times \frac{1}{2} = 25$ lbs.

17. See Fig. to Example 3, page 124 Text-Book. The perpendicular from B upon the direction of $F = CE = r - h$, and the perpendicular EB upon the direction of $W = EB = \sqrt{2rh - h^2}$. Since there is equilibrium, the direction of the resultant of F and W passes through the fixed point B. Taking moments therefore round B we have, by Art. 198,

$$F \times (r - h) = W \times \sqrt{2rh - h^2} \therefore F = W\frac{\sqrt{2rh - h^2}}{r - h}.$$

18. If a diagram be drawn of the arrangement, it will be seen that the vertical from the middle point of the pole (where the weight acts) cuts the ground at a distance of

$3\frac{3}{5}$ feet from the foot of the pole, and that the tension in the cord is at a distance of 8 feet from the lower end. Taking moments round this end we have, therefore,

Tension × 8 = 25 × $3\frac{3}{5}$ $\therefore$ Tension = $11\cdot25$ lbs.

The pressure of the lower end is the resultant of the two forces, the tension in the cord and the weight of the pole, and as these act at right angles, we have, by Art. 170,

Resultant $= \sqrt{11\cdot25^2 + 25^2} = 27\cdot4$ lbs.

NOTE.—There are four forces in the above system—the tension, the weight of the pole, and the reactions of the ground and the wall.

CHAPTER XV.

EXAMINATION QUESTIONS.

1. Explain what is meant by the centre of a system of parallel forces, and state generally the two different methods by which it may be determined.

2. How is the algebraical method of finding the centre of a system of parallel forces divided?

3. When the points of application of a number of parallel forces are in the same line, what is required to be known in order that the centre of the forces should be found?

4. Show how the position of the centre in the foregoing case may be found by calculation.

5. How is the position of a point in a plane determined?

6. When the points of application of a number of parallel forces are in one plane, what must be given in order to determine the position of the centre of the system?

7. Show how the position of the centre in the foregoing case may be found by calculation.

8. When the points of application of a number of parallel forces are not in one plane, what must be given in order to determine the position of the centre of the system?

9. Show how the position of the centre in this case may be calculated.

10. Define the term Centre of Inertia.

11. Define Centre of Gravity.

12. Show that these centres may generally be regarded as coinciding.

13. Give a second definition of the centre of gravity, and show how it is obtained.

14. Define and explain the three states of equilibrium.

15. In each of these states of equilibrium state how a slight displacement of the body affects the position of the

centre of gravity: (*a*), when the body can turn round an axis of suspension; (*b*), when the body rests upon a horizontal plane.

16. A body is placed upon a horizontal plane: what condition must be fulfilled in order that the body may stand? Give the reasons for your answer.

17. If the plane in the last case be inclined, when will the body be on the point of toppling over?

18. State and explain the principles of the stability of bodies.

19. Why is a small boat in danger of being upset when the passengers suddenly stand up?

20. Why cannot a pin remain standing on its point, by itself, on a smooth hard surface?

21. Explain how the pin in the foregoing case may be kept standing.

22. Give some familiar examples of the principle, that in order that a body may stand on its base, the vertical through its centre of gravity must fall within the outline of the base.

23. Show how the centre of gravity of a body may, in some cases, be found by experiment.

24. What are the principles by which the C. G. of a body or a system of bodies may be determined?

25. Show by a geometrical method in each case how to find the C. G. of—

(*a*) A number of heavy particles.
(*b*) A material straight line.
(*c*) A parallelogram.
(*d*) The perimeter of a parallelogram.
(*e*) A triangle.
(*f*) The perimeter of a triangle.
(*g*) Any rectilineal figure.
(*h*) A triangular pyramid.
(*i*) Any pyramid with polygonal base.
(*j*) A cone.

26. Prove that the C. G. of a triangle coincides with the C. G. of three equal heavy particles placed at the angular points of the triangle.

27. Prove that the C. G. of a triangular pyramid coincides with the C. G. of four equal heavy particles placed at the angular points of the pyramid.

28. Show how the position of the C. G. of a system of particles may be found by calculation—

(*a*) When the particles lie in the same straight line.
(*b*) When they are in the same plane.
(*c*) When they are not in the same plane.

29. State, the principle of symmetry, and show how by its means the C. G. of many bodies may be determined.

INTRODUCTORY EXAMPLES TO CHAP. XV.

(*a*) A uniform bar 6 feet long weighs 10 lbs., and to one extremity a weight of 5 lbs. is suspended: upon what point will the bar balance?

The C. G. of the bar is at its middle point since it is uniform (Art. 249), and therefore its weight, 10 lbs., acts at this point. We may therefore imagine the bar weightless, and having two weights, one of 10 lbs. at its middle point, and one of 5 lbs. at the end. The system will be in equilibrium when the resultant of these two forces passes through a fixed point, because then the reaction of the fixed point will be equal and opposite to the resultant of the system. The position of the fulcrum is therefore the point of application of the resultant. This point may be found by Art. 185 or by Art. 202. Choosing the latter method, let x = distance of fulcrum from the end to which the 5 lbs. weight is attached, and take moments round the fulcrum. Then, Art. 202,

$5x = 10(3 - x) \therefore x = 2$ feet from the end.

(*b*) A uniform rod 3 feet long and weighing 1 lb. passes through the centres of three balls weighing 1 lb., 2 lbs., and 3 lbs. respectively, the first having its centre at one end of the rod, the second at the middle point, and the third at the other end: find the C. G. of the system.

Since the rod is uniform its C. G. is at the middle point, and therefore the weight of the rod acts at this point. By Art. 189 the resultant weight $= 1 + (2 + 1) + 3 = 7$ lbs. Taking moments round the end where the 3 lbs. is placed and putting $x =$ distance of C. G. from that end, we have, by Art. 218,

$7x = 1 \times 3 + (2 + 1)1\frac{1}{2} + 3 \times 0 = 7\frac{1}{2} \therefore x = 1\frac{1}{14}$ feet from end where 3 lbs. is placed.

Or:—Taking moments round the middle point, and putting $x =$ distance of C. G. from that point, we have, by Art. 218,

$7x = 3 \times 1\frac{1}{2} + (2 + 1)0 - 1 \times 1\frac{1}{2} = 3 \therefore x = \frac{3}{7}$ foot from middle point $= 1\frac{1}{2} - \frac{3}{7} = 1\frac{1}{14}$ feet from end of rod.

(*c*) At each of five of the angular points of a regular hexagon whose side is 10 inches a weight of 12 ozs. is placed: what is their resultant weight, and what is its point of application?

(*The Student will draw a diagram of the arrangement.*) Let the vacant angular point of the hexagon be called A, and for convenience of reference let A be placed to the right, so that the diagonal drawn from A to the opposite angular point B may be horizontal. Through the middle point of the hexagon draw a vertical axis CD. By Art. 189, the resultant weight $= 5 \times 12 = 60$ ozs. Let $x =$ distance of the point of application of resultant or C.G. of weights from the axis AB. By geometry two of the weights are distant $5\sqrt{3}$ inches above AB, and two $5\sqrt{3}$ inches below it, and one is on the line AB. Taking moments round AB, then Arts. 247, 220,

$60x = 12 \times 5\sqrt{3} + 12 \times 5\sqrt{3} - 12 \times 5\sqrt{3} - 12 \times 5\sqrt{3} = 0.$
Therefore since the moments vanish round AB, the

C. G. is on this line (Art. 204). Let y = distance of C. G. from the axis CD. By geometry two of the weights are distant 5 inches to the right of this line, two are 5 inches to the left, and one (at B) 10 inches to the left. Therefore taking moments round CD,

$60y = 12 \times 5 + 12 \times 5 - 12 \times 5 - 12 \times 5 - 12 \times 10 = -120$ $\therefore y = -2$ inches. Therefore the C. G. of the five bodies is on the line AB and two inches to the left of its middle point.

Or:—If a sixth middle body were placed at A, by the principle of symmetry the C. G. would be at the middle point of AB. Removing the sixth body it is evident that the C. G. of the five remaining bodies moves back along AB towards B. Let x = its distance from the middle point of AB, then, by Art. 202, the moments of the five bodies and of the sixth body round the middle point are equal and opposite. Therefore

$5 \times 12 \times x = 12 \times 10 \therefore x = 2$ inches, the distance of the C. G. from the middle point.

SOLUTIONS OF EXERCISES TO CHAPTER XV. PAGE 165.

1. See Art. 226.

2. For the answer to the first part of the question see Art. 230.

By the Principle of Work (Arts. 132 and 133) the power × length of plane = resistance × height ∴ calling P the power, R the resistance, l the length, and h the height, we have $P = R\frac{h}{l}$. The longer l becomes for the same vertical height, the smaller is the fraction $\frac{h}{l}$, and therefore the smaller the power required to overcome a given resistance. The horse by taking a zigzag path lengthens l while h remains unaltered, and therefore a smaller pull on the load is required.

3. See Art. 226.

4. See Art. 240.

5. See Arts. 238 and 231.

6. (*The Student will draw the diagram.*) Let ABC be the equilateral triangle. Let the particles whose weights are each 2 be placed at the points B and C, and let the weight 1 be placed at A. Then, by geometry, AD, the perpendicular on BC, $= 5\sqrt{3}$ feet. The C. G. of the particles at B and C is at D (Art. 233 or 185), and by the same Arts. the C. G. of the particle 1 at A and the resultant 4 at D is found by dividing AD inversely as these weights. Therefore, as

$4 + 1 : 4 :: 5\sqrt{3} : 4\sqrt{3}$ inches, the distance of the C. G. of the three particles from A.

Or:—Employing Arts. 220, 247 (see also Example 5, page 135 Text-Book). The resultant $= 2 + 2 + 1 = 5$. Let $x =$ distance of C. G. from BC, and take moments round BC $\therefore$

$5x = 1 \times 5\sqrt{3} \therefore x = \sqrt{3}$.

Let $y =$ distance of C. G. from AD, and take moments round AD $\therefore$

$5y = 2 \times 5 - 2 \times 5 = 0$.

Since the moments vanish round AD, the C.G. is in this line (Art. 202); and from the first equation the distance of the C. G. from BC $= \sqrt{3}$ inches $\therefore$ the C. G. lies on AD at a point distant $5\sqrt{3} - \sqrt{3} = 4\sqrt{3}$ inches from the vertex A.

7. By Art. 228 the tower will stand so long as the vertical from its C. G. falls within the base. When the slant height of the tower is 100 feet, it can be readily shown by geometry that the vertical from its centre falls on the circumference of the base, and therefore this is the greatest slant height that the tower can have. Since in this case the slant height to the centre of the tower is 50 feet, and the semi-diameter of the base is 30 feet, the vertical height from the centre to the base $= 40$ feet, and the whole vertical height of the tower $= 2 \times 40 = 80$ feet.

8. (*The Student will draw the diagram.*) Let ABC be the circular table, A, B, and C being points in the circumference equidistant from each other. Join AB, BC, AC; then ABC is an equilateral triangle. Draw AD perpendicular to BC and produce it to meet the circumference in P; bisect AP in G. Then (Art. 249) the weight of the table acts at G. Now if a weight be placed on one of the segments such as BPC, it is evident that its effect in causing rotation round BC will be greatest when it is placed at the point P because then its moment round BC is greatest. But it can be easily shown by geometry that PD = DG, and therefore if a weight equal to the weight of the table be placed at P, the moments round BC will be equal and opposite. Any greater weight placed at P will cause the table to overturn.

9. (*The Student will draw the diagram.*) Let ABC be the corners of the triangular slab. From A draw AD to the middle point of BC, and take G a point in AD so that AG = 2GD. Then (Art. 238) G is the C. G. of the triangle, and the weight of the slab acts at this point. Again (Art. 194) the weight 90 lbs. at G is equivalent to 30 lbs. at A and 60 lbs. at D; and by the same Art. the 60 lbs. at D = 30 lbs. at B and 30 lbs. at C. Therefore since the legs support the whole pressure each of them bears a pressure of 30 lbs.

10. See Example 5, Text-Book, page 165. Let x = distance of C. G. from the side CD, and y = do. from BC. Taking moments round these sides, then (Art. 247 or 220),

$$(3 + 5 + 7 + 9)x = 3 \times 12 + 5 \times 12 \therefore x = 4 \text{ inches.}$$

$$(3 + 5 + 7 \times 9)y = 3 \times 12 + 9 \times 12 \therefore y = 6 \text{ inches.}$$

Therefore the C. G. is 4 inches distant from CD and 6 inches from BC.

11. By Art. 194 the weight 50 lbs. acting at the C. G. of the beam may be resolved into two equal components, one acting at the extremity of the beam 5 feet distant from the C. G., and the other at an equal distance from the C. G. on the other side. Hence if props are placed at these points the pressures on them will be equal.

12. Les x = distance of C. G. from the given extremity of the line. Then taking moments round that extremity we have (Art. 246),

$$(7+4+2+1)x = 1 \times 3 + 2 \times 5 + 4 \times 7 + 7 \times 9 \therefore x = 7\tfrac{3}{7} \text{ ft.},$$
the distance of C. G. from the extremity.

13. Let x = No. of feet in the length of the bar, then $7x$ = weight of the bar in lbs. Since the bar is uniform the weight $7x$ lbs. acts at the middle point. The resultant of the weights passes through the fulcrum since there is equilibrium, the resultant downward weight being then counteracted by the equal and opposite reaction of the fulcrum. Therefore taking moments round the fulcrum, the algebraic sum of the moments is zero (Art. 246), and hence the moments on one side are equal and opposite to those on the other. The distance of the further extremity of the bar from the fulcrum being $(x-3)$ feet, and the distance of the middle point $(\frac{1}{2}x-3)$ feet, therefore

$$20(x-3) + 7x(\tfrac{1}{2}x-3) = 71\tfrac{1}{2} \times 3 \therefore$$
$$7x^2 - 2x = 549 \therefore x = 9 \text{ feet, the length of the bar.}$$

14. Let ABC be the triangle, whose perpendicular height is AL = h, and let DE be drawn parallel to the base BC and cutting the perpendicular AL at M so that AM = $\frac{1}{3}$AL = $\frac{1}{3}h$. Bisect BC in K and join AK. In AK take a point G so that GK = $\frac{1}{3}$AK. Then (Art. 238) G is the centre of gravity of the triangle ABC. Let P, a point in AK, be the C. G. of the triangle ADE, and Q, also in AK, be the C. G. of the space BDEC. Then it can be readily shown from the question, and from Art. 238, that the C. G. of the whole triangle ABC is at a perpendicular distance from BC = $\frac{1}{3}h$, and that the C. G. of ADE is at a perpendicular distance from BC = $(\frac{1}{3}$ of $\frac{1}{3})h + \frac{2}{3}h = \frac{7}{9}h$. Let x = perpendicular distance of C. G. of BDEC from BC. By Euc. VI, 19,

Area of triangle BAC : area of DAE : : $\text{AB}^2 : \text{AD}^2 : : 3^2 : 1$.

Hence if the area ADE be taken = 1, that of ABC = 9, and of BDEC = 8. Taking moments round BC, we have, by Art. 247,

$$1 \times \tfrac{7}{9}h + 8 \times x = 9 \times \tfrac{1}{3}h \therefore x = \tfrac{5}{18}h = \text{distance from BC.}$$

15. In the first position since the books project 3 inches beyond the table the centre of gravity of both is vertically over a point on the table 2 inches from the edge. If now the upper book be pushed 4 inches further on, its C. G. will be 2 inches vertically beyond the edge. Thus in this position the moments of the weights of the books round the edge of the table will be equal and opposite, and the books will therefore remain *just* balanced on the edge.

16. If an eighth body of equal weight to one of the others be placed at the unoccupied angle, then the C. G. of the eight bodies would be at the middle point of the diagonal (Art. 249). Again removing this body, it is evident that the C. G. of the remaining seven bodies will still lie on the diagonal, but more remote from the unoccupied angle. Let x = distance of the C. G. of the seven bodies from the middle point, $\frac{a}{2}$ being the distance of the eighth body from the same point. Let w = weight of each body. Then (Art. 246) the moments round the middle point are equal and opposite, therefore

$7w \times x = w \times \frac{a}{2} \therefore x = \frac{a}{14}$, the distance from the middle point; and $\frac{a}{14} + \frac{a}{2} = \frac{4}{7}a$, the distance from the unoccupied angle.

Or:—We may use the method of Art. 222 or Art. 248, and take moments with respect to each of the three planes whose intersections meet at the unoccupied angle of the cube. Then since three of the particles are in each of these planes, and if as in Art. 222 we take r, r', r'' the distances respectively of the C. G. of the seven bodies from each of the three sides, and if l be the length of the edge of the cube, then by Art. 222,

$$7w \times r = wl + wl + wl + wl \therefore r = \tfrac{4}{7}l.$$
$$7w \times r' = wl + wl + wl + wl \therefore r' = \tfrac{4}{7}l.$$
$$7w \times r'' = wl + wl + wl + wl \therefore r'' = \tfrac{4}{7}l.$$

Hence the C. G. of the seven bodies is equidistant from the three sides of the cube, being at a distance of $\frac{4}{7}l$ from each side. If then planes be drawn parallel respectively to the three sides of the cube from which moments were taken and each plane at a distance of $\frac{4}{7}l$ from the side to which it is parallel, these planes will intersect at a point which is the C. G. of the seven bodies; and it can be easily shown by geometry that this point is on the diagonal of the cube at a distance of $\frac{4}{7}$th of its length from the unoccupied angle.

17. (*The Student will draw the diagram.*) Let ABC be the triangle and let DE be drawn parallel to the base BC, and making the triangle ADE one-fourth of ABC. Then, from geometry, DE is one-half of BC, and the line AK joining A with the middle point of BC is bisected by DE. Let h be the vertical height of the triangle ABC. Then, Art. 238, the C. G. of ABC is at a perpendicular distance from BC $= \frac{1}{3}h$, and the C. G. of ADE $= (\frac{1}{3} \text{ of } \frac{1}{2})h + \frac{1}{2}h = \frac{2}{3}h$ from BC. Let $x =$ distance of C. G. of BDEC from BC. Let the area of ABC be taken $= 1$, then ADE $= \frac{1}{4}$, and BDEC $= \frac{3}{4}$. Then taking moments round BC (Art. 247) we have

$$\tfrac{3}{4} \times x + \tfrac{1}{4} \times \tfrac{2}{3}h = 1 \times \tfrac{1}{3}h.$$

$\therefore \frac{3}{4}x + \frac{1}{6}h = \frac{1}{3}h \therefore \frac{3}{4}x = \frac{1}{6}h \therefore x = \frac{2}{9}h$ from base BC, or $\frac{2}{9}$th of the line bisecting the base measured along the bisector.

18. Circles are proportional to the squares of their diameters, and therefore if the area of the large circle be called A, that of the small circle will be $\frac{1}{4}$A, and that of the remainder will be $\frac{3}{4}$A. Let $x =$ distance of C. G. of the remainder from the centre of the large circle. Take moments round this centre; then since this point is the common C. G. of the small circle and the remaining portion, it follows from

Art. 202 or 246 that the moments round this point are equal and opposite.

> $\therefore \frac{3}{4}A \times x = \frac{1}{4}A \times \frac{1}{4} \therefore x = \frac{1}{12}$th of diameter, the distance of the C. G. of remainder from the centre of the large circle.

19. See Solution to Exercise 9.

20. The middle point of the rod was its C. G. when straight. When the rod is bent the middle point is at the vertex of the right angle formed by the two equal arms, and the point of bisection of the line joining the middle point of these arms is (Arts. 235 and 233) the C. G. of the rod in its new position. Now the line joining the middle points of the arms is the hypotenuse of an isosceles right-angled triangle, each of the equal sides being 10 inches in length: and the distance of the middle point of the hypotenuse from the vertex is, by geometry, equal to half the hypotenuse $= 5\sqrt{2}$ inches.

21. See Exam. 5, page 165 Text-Book. We employ Art. 247. The weight of each side acts at its middle point. The middle point of AB is half of AC distant from BC, and half of BC distant from AC.

> Let x = distance of C. G. from AC, and y = distance from BC. Take moments round AC, then (Art. 247),
>
> $(10 + 8 + 6)x = 10 \times 3 + 6 \times 3 \therefore x = 2$ feet.
>
> Take moments round BC, then
>
> $(10 + 8 + 6)y = 10 \times 4 + 8 \times 4 \therefore y = 3$ feet.
>
> Therefore the C. G. of the perimeter is at a point 2 feet distant from AC and 3 feet from BC.

22. The area of the square = 100, and of the equilateral triangle $= \frac{100}{4}\sqrt{3} = 25\sqrt{3}$. The distance of the C. G. of the triangle from the common side $= \frac{5\sqrt{3}}{3}$ (Art. 238). Let x = distance of C. G. of the whole figure from the common

side. Take moments round the common side; then since the moments of the triangle and the square are of opposite sign, we have, by Art. 247,

$$(100 + 25\sqrt{3})x = 100 \times 5 - 25\sqrt{3} \times \frac{5\sqrt{3}}{3} = 500 - 125$$
$$= 375.$$

$$\therefore x = \frac{375}{100 + 25\sqrt{3}} = \frac{15}{4 + \sqrt{3}},$$ the distance from the side common to the two figures.

23. (*The Student will draw the diagram.*) Let AB be the diagonal of the hexagon and AD, DC, CB the adjacent sides, so that ADCB $= \frac{1}{2}$ a hexagon. Bisect AB in O, and join OD, OC. From O draw OK perpendicular to DC, and DF and CE each perpendicular to AB. Find, Art. 238, the centres of gravity of the triangles DAO and COB and join them. This line is bisected by OK, and the intersection is the C. G. of the two triangles (Art. 233). Call this point M. Find, Art. 238, the C. G. of the triangle ODC; let this point be N. Then divide MN in the ratio 2 : 1 in the point G. G is the C. G. of the whole figure (Art. 233).

24. Let x = distance of C. G. from the middle point. Take moments round this point; then, Art. 246,

$$(7 + 5 + 3 + 1)x = 7 \times 12 + 5 \times 4 - 3 \times 4 - 1 \times 12$$

$\therefore 16x = 80 \therefore x = 5$ inches from middle point.

25. Since the square is a material lamina of uniform thickness, the weights are proportional to the areas. The area of the triangle taken away $= 25$, and of the three remaining $= 75$. When the triangle is replaced, the C. G. of the whole is the middle point of the square. Therefore, Art. 246, the moments of the triangle and the remaining portion of the square round this point are equal and opposite (Art. 246 or 202). The distance of the C. G. of the triangle from the centre of the square $= \frac{2}{3} \times 5 = \frac{10}{3}$ feet (Art. 238).

Let $x =$ the distance of the C. G. of remainder from the centre $\therefore 75x = 25 \times \frac{10}{3} \therefore x = 1\frac{1}{9}$ feet, the distance from the centre.

26. Proceeding as in last solution, let $x =$ distance of remaining part from the centre, then

$$75x = 25 \times 2\tfrac{1}{2} \therefore x = \tfrac{5}{6} \text{ foot, the distance from the centre.}$$

27. The area of the square cut out $= 1$ square inch, and of the remainder $= 15$ square inches. The distance of the C. G. of the small square from the centre of the large square is $\frac{3}{2}\sqrt{2}$ inches. Let $x =$ the distance of the C. G. of remaining part from the centre of the square. Then taking moments round the centre, we have, by Art. 246,

$$15x = 1 \times \tfrac{3}{2}\sqrt{2} \therefore x = \tfrac{1}{10}\sqrt{2} \text{ inches, the distance from the centre.}$$

28. The *surface* in this case means an indefinitely thin material lamina bent into a conical shape. If this surface be cut from the vertex to a point in the circumference and then flattened out, it will form a sector of a circle. This sector may be regarded as made up of an infinite number of triangles having their bases on the circumference and their vertices at the apex of the sector. The C. G. of each of these triangles is at a distance from the base of $\frac{1}{3}$rd the perpendicular height (Art. 238), and therefore if the sector be once more rolled into the surface of a cone, its C. G. will lie in a plane parallel to the base of the cone, and at a distance from the base $= \frac{1}{3}$rd the perpendicular height of the cone. Hence, by Art. 249, the C. G. of the surface will be the intersection of this plane by the perpendicular height.

29. The weights are proportional to their cubic contents. The solid contents of the cylinders are proportional to the products of their heights into the squares of their respective diameters $\therefore$ the cubic content of the first cylinder is proportional to $2^2 \times 8 = 32$, and of the second to $3^2 \times 6 = 54$, and the sum of their contents to 86. The distance of the C. G.

of the first cylinder from the common surface of the two cylinders $= 4$, and the distance of the C. G. of the second $= 3$. Let $x =$ distance of C. G. of the whole mass from the common surface, then taking moments with respect to this surface, we have (Art. 222),

$86x = 54 \times 3 - 32 \times 4 \therefore x = \frac{17}{43}$ inch from the common surface.

30. For convenience of reference let the octagon be arranged so that the weights 1 and 2 and the weights 6 and 5 will be at the extremities of *horizontal* sides, and the weights 3 and 4 as also 8 and 7 at the extremities of *vertical* sides. Through the centre point of the octagon draw two axes, one horizontal and the other vertical. Take moments round these axes, calling moments above the horizontal line positive, and those below negative, those to the right of the vertical positive, and those to the left negative. The distance of each of the weights 1, 2, 6, 5 from the horizontal line is, by geometry, $\frac{1}{2}a(1+\sqrt{2})$, and that of each of the weights 8, 3, 4, 7 is $\frac{1}{2}a$. The distance of each of the latter from the vertical is $\frac{1}{2}a(1+\sqrt{2})$, and of the former is $\frac{1}{2}a$. Take moments round the horizontal axis. (See Exam. 5, page 165 Text-Book), then, by Art. 247,

$$\begin{aligned} 36x &= 8 \times \tfrac{1}{2}a + 1 \times \tfrac{1}{2}a(1+\sqrt{2}) + 2 \times \tfrac{1}{2}a(1+\sqrt{2}) + 3 \times \tfrac{1}{2}a \\ &\quad - 4 \times \tfrac{1}{2}a - 5 \times \tfrac{1}{2}a(1+\sqrt{2}) - 6 \times \tfrac{1}{2}a(1+\sqrt{2}) - 7 \times \tfrac{1}{2}a \\ &= -8 \times \tfrac{1}{2}a(1+\sqrt{2}) = -4a(1+\sqrt{2}) \therefore x = \frac{a}{9}(1+\sqrt{2}) \end{aligned}$$

below the horizontal.

Taking moments round the vertical axis,

$$\begin{aligned} 36y &= 2 \times \tfrac{1}{2}a + 3 \times \tfrac{1}{2}a(1+\sqrt{2}) + 4 \times \tfrac{1}{2}a(1+\sqrt{2}) + 5 \times \tfrac{1}{2}a \\ &\quad - 1 \times \tfrac{1}{2}a - 8 \times \tfrac{1}{2}a(1+\sqrt{2}) - 7 \times \tfrac{1}{2}a(1+\sqrt{2}) - 6 \times \tfrac{1}{2}a \\ &= -4a(1+\sqrt{2}) \therefore y = \frac{a}{9}(1+\sqrt{2}) \end{aligned}$$

to the left of the vertical.

Therefore the distance from the centre =

$$\sqrt{\left[\frac{a}{9}(1+\sqrt{2})\right]^2+\left[\frac{a}{9}(1+\sqrt{2})\right]^2}=\sqrt{\frac{2a^2}{81}(1+2\sqrt{2}+2)}$$

$$=\sqrt{\frac{a^2}{81}(6+4\sqrt{2})}=\frac{a}{9}(2+\sqrt{2}).$$

31. Arrange the hexagon so that the weights 1, 2, and 5, 6 are at the extremities of horizontal sides. Draw through the centre of the hexagon horizontal and vertical axes and take moments round these axes as in the solution to the last exercise. Then round horizontal axis we have

$21x = 1 \times \frac{1}{2}a\sqrt{3} + 2 \times \frac{1}{2}a\sqrt{3} - 4 \times \frac{1}{2}a\sqrt{3} - 5 \times \frac{1}{2}a\sqrt{3} =$
$-3a\sqrt{3} \therefore x = \frac{a}{7}\sqrt{3}$, the distance of the C. G. below the horizontal axis.

Round vertical axis we have

$21y = 2 \times \frac{1}{2}a + 3 \times a + 4 \times \frac{1}{2}a - 5 \times \frac{1}{2}a - 6 \times a - 1 \times \frac{1}{2}a =$
$-3a \therefore y = \frac{a}{7}$, the distance to the left of the vertical.

Therefore the distance of the C. G. from the centre of the hexagon $= \sqrt{\left(\frac{a}{7}\sqrt{3}\right)^2+\left(\frac{a}{7}\right)^2} = \sqrt{\frac{3a^2}{49}+\frac{a^2}{49}} = \frac{2a}{7}.$

32. See Solutions to Exercises 25 and 26.

Let $x =$ distance of C. G. of remaining part from the centre of the square. The distance of the C. G. of the triangle from the centre is (Art. 238) $= \frac{2}{3}$ of $\frac{1}{2}a = \frac{1}{3}a$. Taking moments round the centre,

$3x = 1 \times \frac{1}{3}a \therefore x = \frac{1}{9}a$, the distance of C. G. from centre.

33. Since the whole cone and the cone cut off are similar solids they are proportional to the cubes of their heights. Therefore the contents and consequently the weights of the

cones are proportional to 64 and 8 respectively, and the contents of the frustum to $64 - 8 = 56$. By Art. 245 the distance of the C. G. of the large cone from its base $= 1$ inch, and the distance of the C. G. of the small cone from the same base $= (\frac{2}{4} + 2)$ inches $= 2\frac{1}{2}$ inches. Let $x =$ distance of C. G. of frustum from the base, and take moments round the latter $\therefore$

$$56x + 8 \times 2\tfrac{1}{2} = 64 \times 1 \therefore x = \tfrac{11}{14} \text{ inch, the distance from the base.}$$

34. See Exam. 3, p. 145 Text-Book. Let R be the reaction at the lower extremity and R′ at the upper. Resolve vertically and horizontally and take moments round the lower extremity. Then

$$R - W = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$T - R' = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$W \times 3\tfrac{1}{2} = R' \times 24 \quad . \quad . \quad . \quad . \quad (3)$$

From (3) $R' = 14\frac{7}{12}$, and from (2) $R' = T \therefore T = 14\frac{7}{12}$ lbs.

35. See Exam. 5, p. 165 Text-Book. Let $x =$ the distance of the C. G. from the side which has at its extremities the weights 5 and 9, and let $y =$ distance from side having weights 3 and 5. Taking moments round these sides, we have

$$18x = 3 \times 4 + 1 \times 4 = 16 \therefore x = \tfrac{8}{9} \text{ ft.}$$

$$18y = 1 \times 4 + 9 \times 4 = 40 \therefore y = 2\tfrac{2}{9} \text{ ft.}$$

CHAPTER XVI.

EXAMINATION QUESTIONS.

1. What is a Machine?

2. State the general equation which applies to all machines moving uniformly, and show how it is obtained.

3. What are the Simple Machines? How may they be classified?

4. What is the *mechanical advantage* of a machine?

5. What is a Lever?

6. Classify levers, and give examples of each class.

7. Give an example of each kind of lever in the human body.

8. State the equation for the lever which expresses the relation between the power and the resistance when in equilibrium.

9. Show how this equation is obtained: (*a*), when the lever is at rest under the action of the forces acting on it; (*b*), when it is moving uniformly under their action.

10. Sketch a combination of levers, and state the object of thus combining them.

11. State and prove the rule for a combination of levers when there is equilibrium.

12. Describe the Common Balance.

13. What are the requisites of a good balance? Define each.

14. Upon what does the *accuracy* of a balance depend?

15. Show that it is requisite that the beam and the supports of the scales should rest on knife edges.

16. State and explain the effects that would be produced:—

    (*a*) If the scale pans were rigidly connected with the beam;

    (*b*) If when the beam is in equilibrium a scale pan were pushed inwards or outwards.

17. State the conditions on which the *stability* of the balance depends.

18. Show that the C. G. of the beam and scales should be below the axis of suspension.

19. Show that the axes of suspension of the scales should be in the same plane with the axis of suspension of the beam.

20. Show that the beam should be rigid.

21. Upon what does the *sensibility* of the balance depend?

22. How are the conditions of sensibility secured in a well-constructed balance?

23. Show that stability and sensibility are to some extent incompatible.

24. How may the equality of the arms of a balance be tested?

25. Show how by an experimental method the true weight of a body may be obtained with a false balance.

26. If the apparent weight of a body in each scale of a false balance be given, state and prove the rule by which the true weight may be calculated.

27. Describe the Common Steelyard, and point out its advantages.

28. Show how the beam may be graduated: (*a*), by an experimental method; (*b*), by calculation.

29. Describe the Danish Steelyard, and show how it may be graduated: (*a*), by experiment; (*b*), by calculation.

30. Compare the graduations of the Common and the Danish Steelyard.

31. Describe the Wheel and Axle.

32. Show that the wheel and axle may be regarded as a system of levers.

33. State the rule for the wheel and axle, and show how it is obtained: (*a*), when at rest; (*b*), when moving uniformly.

34. Give practical examples of the wheel and axle.

35. Describe the Differential Wheel and Axle, and explain the principle upon which it is constructed.

36. State and prove the equation for the Differential Wheel and Axle which expresses the relation between the power and resistance when in equilibrium.

37. State and prove the rule for *toothed* wheels when there is equilibrium.

38. State the rule for a combination of wheels and axles.

39. Describe the Inclined Plane used as a Simple Machine and define the terms *height, length,* and *inclination of the plane.*

40. State and derive the relation between the power, the weight, and the reaction of the plane when there is equilibrium with an inclined plane :—

(*a*) When the power acts parallel to the length ;
(*b*) When it acts parallel to the base ;
(*c*) When its direction makes a given angle with the perpendicular to the plane.

41. Show how the foregoing results may be obtained in several ways.

42. When two bodies support each other on two inclined planes of equal height, state and prove the relation between them.

43. Show that the Wedge is a form of the Inclined Plane.

44. Give examples of the practical application of the principle of the wedge.

45. Give a description of the Screw.

46. Show that the screw is a form of the inclined plane.

47. Obtain the equation for the screw when there is equilibrium.

48. Describe the Differential Screw and explain the principle upon which it depends.

49. Obtain the equation for the Differential Screw.

50. Describe the different parts of the simple machine called the Pulley.

51. What is the use of the little wheels in pulleys ?

52. Determine the relation of the power to the weight when there is equilibrium with a single movable pulley :—

(*a*) When the cords are parallel ;
(*b*) When inclined at a given angle.

53. Show how the relation between the power and the weight when in equilibrium may be determined—(1) neglecting the weights of the pulleys; (2), taking the weights into account—in each of the following cases:—

(*a*) In a system of pulleys with a single string;
(*b*) In a system with separate strings, where each block is supported by two tensions;
(*c*) In a system where each block is supported by three tensions;
(*d*) In a system where each pulley has a separate cord the end of which is attached to the weight.

54. Obtain in each of the foregoing cases the general equation which expresses the relation of P to W.

55. Determine in each case the pressure on the fixed support.

56. Show how the results required in question 53 may be obtained by the Principle of Work.

INTRODUCTORY EXAMPLES TO CHAP. XVI.

(*a*) In a lever of the first kind the fulcrum is 2 ft. distant from the power, and ½ ft. distant from the resistance: if the latter be 100 lbs. what is the power that will produce equilibrium?

The arms of the power and weight are respectively 2 ft. and ½ ft., and, by Art. 255,

$P \times 2 = 100 \times \frac{1}{2} \therefore P = 25$ lbs.

(*b*) In a lever of the second kind the power is 4 ft. distant from the weight, and the weight 1 ft. from the fulcrum; if the power be 10 lbs., what weight will it support?

The arms of the power and weight are the distances from the fulcrum to the directions of the power and weight respectively, therefore arm of $P = 5$ ft., and arm of $W = 1$ ft. Hence, Art. 255,

$10 \times 5 = W \times 1 \therefore W = 50$ lbs.

(*c*) In a lever of the third kind the distance from the weight to the power is 3 feet, and from the power to the fulcrum is 5 feet: what power is required to support a weight of 20 lbs.?

The arm of $P = 5$ ft., and the arm of $W = 8$ ft., therefore, Art. 255,
$P \times 5 = 20 \times 8 \therefore P = 32$ lbs.

(*d*) In a wheel and axle, the circumference of the wheel $= 10$ ft. and that of the axle $= \frac{1}{2}$ ft.: what power will support a weight of 1 cwt.?
$P \times 10 = 112 \times \frac{1}{2} \therefore P = 5 \cdot 6$ lbs.

(*e*) What power, acting parallel to the length of an inclined plane whose height is 6 ft. and base 8 ft., will sustain a weight of 120 lbs.?

By geometry the length of the plane $= 10$ feet; and, by Art. 279,
$P : W :: h : l \therefore P : 120 :: 6 : 10 \therefore P = 72$ lbs.

(*f*) The circumference of the circle described by the power in a screw is 10 ft., and the distance between the threads is 1 inch: what power will equilibrate 1 ton?
By Art. 287, $P \times C = W \times d \therefore P \times 10 = 2240 \times \frac{1}{12} \therefore P = 18\frac{2}{3}$ lbs.

(*g*) In a system of four movable pulleys where each movable block is supported by a separate cord the end of which is attached to a beam, the power is 20 lbs.; find the weight (*a*), when the weights of the blocks are neglected; (*b*), when they are taken into account, each weighing $\frac{1}{2}$ lb.

See Art. 294. (*a*) The tension under the first movable pulley $= 20 \times 2 = 40$; under the second $= 40 \times 2 = 80$; under the third $= 80 \times 2 = 160$; under the fourth $= 160 \times 2 = 320$ lbs., the required weight.
Or:—Art. 294, $W = P \times 2^n = 20 \times 2^4 = 320$ lbs.

(*b*) The tension under the first movable pulley $= 20 \times 2 - \frac{1}{2} = 39\frac{1}{2}$; under the second $= 39\frac{1}{2} \times 2 - \frac{1}{2} = 78\frac{1}{2}$; under the third $= 78\frac{1}{2} \times 2 - \frac{1}{2} = 156\frac{1}{2}$; under the fourth $= 156\frac{1}{2} \times 2 - \frac{1}{2} = 312\frac{1}{2}$ lbs., the required weight.

SOLUTIONS OF EXERCISES TO CHAP. XVI. PAGE 190.

1. See Art. 272.
2. See Art. 263.
3. See Art. 283. P : Q :: 15 : 20 :: 3 : 4.
4. See Arts. 266–270.

5. By Art. 194, we divide the resultant weight of 112 lbs. into two others acting at distances 4 ft. and 5 ft. respectively.

Hence as $(5+4):4::112:49\frac{7}{9}$ lbs., the weight borne by the man 5 feet distant from the cask.

And $(5+4):5::112:62\frac{2}{9}$ lbs., the weight borne by the other.

6. Dividing 10 ft. in the ratio of 12 to 18, we find that the fulcrum is distant 4 ft. from the end where the 18 lbs. is attached.

Or:—Let x = distance from end where 18 lbs. is attached.

Then, Art. 255, $18x = 12(10-x) \therefore x = 4$ feet.

7. The oar of a boat is a lever of the second kind, the weight acting between the power and the fulcrum. The fulcrum is where the oar touches the water, the power where the oar is grasped by the hand, and the weight at the rowlock.

The handle of a pump is a lever of the first kind, the fulcrum being between the power and the weight. The weight acts at the extremity where the piston rod is attached, the power where the hand is applied, and the fulcrum is the axle on which the handle turns.

A crowbar may be used so as to be an example of any of the kinds of lever. When it is driven into the ground under a stone and pulled backwards in order to lift the stone, it is a lever of the first kind, the fulcrum being between the weight and the power. The weight is the stone acting at one end, the power acts at the other, and the edge of the hole on which the bar rests is the fulcrum.

A wheelbarrow when lifted by the handles is an example of a lever of the second kind, the weight acting between the fulcrum and the power. The power acts at the handles, the weight at the box of the wheelbarrow, and the fulcrum is at the axle of the wheel.

A balance is a lever of the first kind with equal arms, the fulcrum being intermediate between the points of application of the two forces acting on the beam.

A pair of tongs is a double lever of the third kind, the power acting at a point between the weight and the fulcrum. The resistance is the pressure exerted at the feet, the fulcrum is at the rivet of the legs, and the power acts where the hand grasps the tongs.

A pair of scissors is a double lever of the first kind, the fulcrum being between the power and the resistance. The power acts at the handles, the resistance where the body is cut by the blades, and the fulcrum is at the rivet.

Examples of each kind of levers in the human body are given in Art. 254.

8. See Art. 293. The sketch is similar to the first figure with an additional movable and an additional fixed pulley.

(*a*) $W = P \times 2n \therefore 160 = P \times 2 \times 4 \therefore P = 20$ lbs.

(*b*) $W + w = P \times 2n \therefore 160 + 4 = P \times 2 \times 4 \therefore P = 20\frac{1}{2}$ lbs.

9. The sketch is like the second figure of Art. 293, with the last pulley left out. The successive tensions are formed as in Art. 294.

(*a*) Tension above lowest pulley $= \frac{100}{2} = 50$; above 2nd $= \frac{50}{2} = 25$; above 1st $= \frac{25}{2} = 12\frac{1}{2}$ lbs., the required power.

(*b*) Tension above lowest pulley $= \frac{100+1}{2} = 50\frac{1}{2}$; above 2nd $= \frac{50\frac{1}{2}+1}{2} = 25\frac{3}{4}$; above 1st $= \frac{25\frac{3}{4}+1}{2} = 13\frac{3}{8}$ lbs., the required power.

10. By Art. 293, $W = P \times 2n \therefore 48 = 6 \times 2n \therefore$

$n = \frac{48}{6 \times 2} = 4$, the number of pulleys.

11. See Art. 281.

12. See Art. 296.

13. See Art. 280. $P : W :: 6 : 8 \therefore P = 84$ lbs.

And $R : W :: 10 : 8 \therefore R = 140$ lbs.

14. Since the goods are placed in the scale attached to the shorter arm, the weight of the goods is greater than that of the weights when there is equilibrium. Hence the shopkeeper loses in selling the goods.

Let x = the weight of goods in the scale attached to the shorter arm which balances 1 cwt. placed in the other scale, then, by Art. 255,

$x \times 11 = 1$ cwt. $\times 12 \therefore x = 1\frac{1}{11}$ cwt. Hence goods weighing $1\frac{1}{11}$ cwt. are sold for 1 cwt., therefore the shopkeeper loses $\frac{1}{11}$ cwt. in every cwt. sold, or $10\frac{2}{11}$ lbs. per cwt.

15. The weight is double the power. See Art. 291.

16. See figure of Art. 294, omitting the last movable pulley.

$W = P \times 2^n = 20 \times 2^3 = 160$ lbs.

NOTE.—The pulleys might be arranged in other ways, for example according to the arrangement of Art. 295 or of 296; and a different weight would be supported in each case.

17. Since $W = 8P$, by the Principle of Work P will move through 8 times the space moved by W. Therefore space required = 8 feet.

18. See Arts. 260–262.

19. The arms are inversely as the weights that produce equilibrium, and are therefore as 10 : 12 or as 5 : 6. When the body appears to weigh only 10 lbs. it must be attached to the shorter arm. Let it now be placed in the scale suspended from the longer arm, and let x be the weight in the other scale that produces equilibrium. Then, Art. 255,

$$x \times 5 = 12 \times 6 \therefore x = 14 \cdot 4 \text{ lbs.}$$

Or:—Since by Art. 264, the square of the true weight = the product of the apparent weights, it follows that 12^2 = the product of the apparent weights; and since one of these is 10, the other is $\frac{144}{10} = 14 \cdot 4$ lbs.

20. Let x = the distance of the fulcrum from the middle point, then, Art. 255, the resultant of all the weights passes through the fulcrum, and by Art. 246, the algebraic sum of the moments of the weights round the fulcrum is equal to zero, therefore the sum of the two moments on one side must be equal to the moment on the other. Therefore

$4(6 + x) + 6x = 10(6 - x) \therefore x = 1\frac{4}{5}$ feet, the distance from the centre of the bar.

21. See Solution to Ex. 19. When the body appears to weigh 24 ozs. it must be attached to the longer arm. The arms are as 5 : 6. Let x = the required weight, then, by Art. 255,

$$x \times 6 = 20 \times 5 \therefore x = 16\frac{2}{3} \text{ ozs.}$$

22. See Solution to Ex. 20. Let x = the distance of the counterpoise from the fulcrum, and take moments round the fulcrum. Then, by Art. 246,

$6 \times 4 = 2 \times 1 + 1 \times x \therefore x = 22$ inches, the distance from fulcrum.

23. The weight of the bar acts at its middle point. Let x = the distance of the fulcrum from the extremity where the

8 lbs. is attached, and take moments round the fulcrum. Then, by Art. 246,

$10(3 - x) = 8 \times x \therefore x = 1\frac{2}{3}$ ft., the distance from the end of the bar.

The pressure on fulcrum = resultant of weights (Art. 255) = $10 + 8 = 18$ lbs.

24. Since the beam by itself balances on a fulcrum 2 ft. distant from the thicker end, the C. G. of the beam is 8 ft. distant from the smaller. When the weight of 120 lbs. is attached the fulcrum is 2 ft. from the smaller end, and is therefore 6 ft. from the point where the weight of the beam acts. Calling this weight W and taking moments round the fulcrum, we have

$W \times 6 = 120 \times 2 \therefore W = 40$ lbs.

25. (*a*) The power required = $100 \times 10 = 1000$ lbs.

(*b*) The power required to overcome friction and resistance of air = 1000 lbs., and to draw the train up the incline = $\frac{1}{100}$th of its weight = $\frac{100 \times 2240}{100} = 2240$ lbs. $\therefore$ the power required = $1000 + 2240 = 3240$ lbs.

Note.—The power required to draw the train up the incline is obtained from Art. 279.

$P : W :: \text{Height} : \text{Length} \therefore P : 100 \times 2240 :: 1 : 100.$

26. See last solution. A force = $\frac{1}{10}$th load is required to overcome friction, and a power = $\frac{1}{10}$th load to draw it up the incline of 1 in 10 (Art. 279). Therefore the whole force $= \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$th load $= \frac{2240}{5} = 448$ lbs.

27. See Solution to Exam. 5, page 190 Text-Book.

28. By the Principle of Work, $P \times 120 = 2240 \times 3 \therefore P = 56$ lbs.

29. By Art. 258, $P \times 4 \times 7 \times 5 = W \times 1 \times 2 \times 2 \therefore \frac{W}{P} = \frac{140}{4} = 35$, the mechanical advantage.

30. In selling the first cwt. let the goods be placed in the scale attached to the shorter arm, and let x be their true weight. Then $x \times 7 = 1$ cwt. $\times 8 \therefore x = 1\frac{1}{7}$ cwt., and therefore the seller loses $\frac{1}{7}$th of a cwt., since he sells $1\frac{1}{7}$ cwt. for 1 cwt.

In selling the second cwt. let the goods be placed in the scale attached to the longer arm, and let y be their true weight. Then $y \times 8 = 1$ cwt. $\times 7 \therefore y = \frac{7}{8}$ cwt., and therefore the seller gains $\frac{1}{8}$ cwt., since he sells $\frac{7}{8}$ cwt. for 1 cwt.

Therefore, in the sale of 2 cwts., the seller loses $(\frac{1}{7} - \frac{1}{8}) = \frac{1}{56}$ cwt. = 2 lbs.—that is, he loses 1 lb. per cwt.

31. By Art. 272, $P \times C = W \times c \therefore 120 \times 12 = W \times 1\frac{1}{2} \therefore W = 960$ lbs.

32. By geometry the base $= \sqrt{25^2 - 7^2} = 24$.

(*a*) $P : W ::$ height : length $\therefore P : 2240 :: 7 : 25 \therefore P = 627\frac{1}{5}$ lbs.

(*b*) $P : W ::$ height : base $\therefore P : 2240 :: 7 : 24 \therefore P = 653\frac{1}{3}$ lbs.

33. By Art. 279, Height : Length :: Power : Weight
:: 10 : $16\frac{2}{3}$
:: 3 : 5.

34. Let $x =$ the weight on the shorter plane, then, by Art. 283, $x : 100 :: 15 : 25 \therefore x = 60$ lbs.

35. By Art. 287, $P \cdot C = W \cdot d \therefore P \times 15 = 100 \times \frac{1}{4} \therefore P = 1\frac{2}{3}$ lbs.

36. By Art. 287, $P \cdot C = W \cdot d$. The diameter of the circle described by the power being 120 inches, $C = 120 \times 3{\cdot}14159 \therefore P = \frac{W \cdot d}{C} = \frac{1120 \times \frac{1}{2}}{120 \times 3 \cdot 14159} = 1{\cdot}5$ lbs. nearly.

37. See Arts. 252 and 289. The mechanical advantage in the screw is the ratio $W:P$, which is the same as the ratio $C:d$—that is, the ratio of the circumference of the circle described by the power to the distance between two threads of the screw.

38. The diameter of the wheel = 30 inches. As 7 : 22 :: 30 : $\frac{660}{7}$ inches, the circumference of the wheel. Hence in 7 revolutions the power moves through $\frac{660}{7} \times 7 = 660$ inches; and, by question, the weight rises through $5\frac{1}{2}$ ft. = 66 inches. Therefore, by the Principle of Work,

$$P \times 660 = 30 \times 66 \therefore P = 3 \text{ lbs.}$$

39. In order to sustain the weight, the man has to pull on the cord with a force = the weight = 112 lbs. (Art. 291). The pressure on the ground due to his weight is evidently lessened by the amount of the pull, therefore this pressure $= 168 - 112 = 56$ lbs.

40. The strain on the beam (Art. 297) is equal to the sum of the tensions that act on the beam; therefore the strain in this case $= 112 + 112 = 224$ lbs.

41. By Art. 294, the pull required to sustain the weight of 112 lbs. $= \frac{W}{2^n} = \frac{112}{2^3} = 14$ lbs. The pressure on the ground is lessened by this amount, therefore the required pressure $= 168 - 14 = 154$ lbs.

42. See Art. 294. The tension above the lowest pulley $= \frac{112 + 2}{2} = 57$; above the second $= \frac{57 + 2}{2} = 29\frac{1}{2}$; above the first $= \frac{29\frac{1}{2} + 2}{2} = 15\frac{3}{4}$, which is the pull required to sustain the weight. Therefore the pressure on the ground $= 168 - 15\frac{3}{4} = 152\frac{1}{4}$ lbs.

43. By Art. 293, $W = P \times 2n = 50 \times 2 \times 4 = 400$ lbs.

44. The whole weight supported $= 112 + 4 = 116$ lbs. By Art. 293, $W = P \times 2n \therefore P = \frac{W}{2n} = \frac{116}{2 \times 4} = 14\frac{1}{2}$ lbs.

45. See Art. 295. The tension above the lowest pulley $=\frac{135}{3}=45$; above the second $=\frac{45}{3}=15$; above the first $=\frac{15}{3}=5$ lbs., the required force.

46. The mechanical advantage (Art. 252) $=135:5=27$.

47. See Art. 295. The tension below the first pulley $=20\times 3-3=57$; below the second $=57\times 3-3=168$; below the third $=168\times 3-3=501$ lbs., the weight required.

48. The whole weight = 200 lbs., and this is to be raised by a power of 20 lbs. Now, in the arrangement of Art. 293, $W=P\times 2n$. Substituting the values of W and P, we have, $200=20\times 2n$; so that with the system of Art. 293, if we use 5 movable pulleys the power of 20 lbs. will sustain the weight of 200 lbs. By the Principle of Work, when the system is moving uniformly the power must move through 10 times the space moved by the weight, and therefore the distance through which the power acts will be the same as if it were applied directly to raise the weights one by one.

49. See Fig. of Art 296. Since there are five movable pulleys, there are in all six pulleys, the upper one being fixed, and there are six cords connected with the weight. In the cord passing round the lowest pulley there is no tension since there is no power; therefore the tension of the first cord connected with the weight = 0. The first movable pulley weighing $3\frac{1}{2}$ ozs., this is the tension throughout the cord passing over the second pulley; therefore the tension of the second cord joined to the weight $=3\frac{1}{2}$ ozs. Since two tensions, each equal to $3\frac{1}{2}$ ozs., act on the second pulley, it is pulled down with a force of $3\frac{1}{2}\times 2=7$ ozs.; and as the weight of the pulley $=3\frac{1}{2}$ ozs., the third cord has a tension of $3\frac{1}{2}\times 2+3\frac{1}{2}=10\frac{1}{2}$ ozs. Similarly the fourth cord has a tension of $10\frac{1}{2}\times 2+3\frac{1}{2}=24\frac{1}{2}$ ozs.; the fifth a tension $=24\frac{1}{2}\times 2+3\frac{1}{2}=52\frac{1}{2}$ ozs.; and the sixth a tension $=52\frac{1}{2}\times 2+3\frac{1}{2}=108\frac{1}{2}$ ozs. The weight is supported by all these tensions, therefore, since there is equilibrium, the weight $=3\frac{1}{2}+10\frac{1}{2}+24\frac{1}{2}+52\frac{1}{2}+108\frac{1}{2}=199\frac{1}{2}$ ozs.

50. By Art. 258, $P \times 12 \times 25 = 150 \times 1 \times 2 \therefore P = 1$ lb.

51. By Art. 272 $(12 \times 42) \times 7\frac{1}{2} = W \times 1\frac{1}{4} \therefore W = 3024$ lbs.

52. This is a lever of the second kind. The arm of the power = 8 feet, and the arm of the weight = 3 feet. By Art. 255,

$P \times 8 = 40 \times 3 \therefore P = 15$ lbs.

Pressure on fulcrum = resultant of the forces = W − P, since they act in opposite directions $= 40 - 15 = 25$ lbs.

53. The oar is a lever of the second kind, the fulcrum being at the end which touches the water, and the weight acts at the rowlock. The arm of power is, therefore, 12 feet, and the arm of weight 9 feet. By Art. 255,

$30 \times 12 = W \times 9 \therefore W = 40$ lbs.

54. In this windlass the power is the push applied at such a point of the handle that the arm of power = 36 inches. The resistance acts along the centre of the cord which passes round the axle, and as the radius of the axle is 3 inches, and half the thickness of the rope $\frac{1}{2}$ inch, the arm of the weight = $3\frac{1}{2}$ inches. Then, Art. 272,

$P \times R = W \times r \therefore 120 \times 36 = W \times 3\frac{1}{2} \therefore W = 1234\frac{2}{7}$ lbs.

55. By Art. 287, $P \times C = W \times c$. The diameter of the circle described by the power being 12 feet, the circumference $= 12 \times 3 \cdot 14159$; and as there are 150 threads in 1 ft., the distance between two threads $= \frac{1}{150}$ ft. Therefore,

$P \times 12 \times 3 \cdot 14159 = 2240 \times \frac{1}{150} \therefore P = \cdot 396$ lb.

56. By Art. 276, $3 \times 5 \times 5 \times 5 \times 5 = W \times 1 \times 1 \times 1 \times 1$ $\therefore W = 1875$ lbs.

57. By Art. 274, $50 \times 10 = W \times \frac{1 - \cdot 9}{2} \therefore W = 10000$ lbs.

58. Since the weight = 200 times the power, then, by the Principle of Work, the power must move through 200 times the space moved by the weight. Therefore space required $= 1 \times 200 = 200$ inches $= 16\frac{2}{3}$ feet.

[*] CHAPTER XVII.

EXAMINATION QUESTIONS.

1. What is Friction?

2. What reactions are possible (*a*), between smooth surfaces; (*b*), between rough surfaces?

3. Explain what is meant by the angle of friction for two bodies.

4. Define *statical* friction.

5. Define *kinetic* friction, and mention its different kinds.

6. State the Laws of friction, and explain how they have been obtained.

7. What are the limitations to the truth of these laws?

8. Explain what is meant by the *coefficient of friction* for two bodies.

9. State and prove the relation which exists between the coefficient of friction and the angle of friction.

10. How may the angle of friction be determined by experiment, and how may the coefficient of friction be then calculated?

11. How may the coefficient of friction be determined by experiment, and how may the angle of friction be then calculated?

12. Show that when a surface is inclined at such an angle that a body resting upon it is on the point of sliding, the angle of inclination is the angle of friction for the two surfaces.

13. What conditions affect the value of the coefficient of friction for the same two surfaces?

14. Show that friction always helps the weaker force.

15. If the weight of a body be W, and the coefficient of friction be μ, determine the friction (*a*), when the body is on the point of moving on a horizontal plane; (*b*), when in motion on the plane.

16. A body whose weight is W, is sliding down a rough inclined plane whose inclination is i; if the coefficient of friction be μ, determine the inclination.

17. How may the relations between space, time, and velocity be determined in the case of bodies moving upon *rough* inclined planes?

18. State and prove the equation which expresses the relation between the power and the resistance in the Lever, taking friction into account.

19. State the similar equation for the Wheel and Axle.

20. Investigate the relation between the power and the weight for the general case of equilibrium on a rough Inclined Plane, and from it derive the equations which express the relations for the other cases.

21. Determine the relations between the power and the resistance on a rough inclined plane (*a*), when the body is on the point of moving up the plane; (*b*), when on the point of moving down.

22. A body whose weight is W rests on a horizontal plane and is on the point of moving under the action of a force P which acts at an inclination θ to the horizontal plane, find the value of P, the coefficient of friction being μ.

23. Determine the most advantageous angle at which a force may be exerted on a body placed on a horizontal plane.

24. Determine the most advantageous angle at which a force may be exerted on a body placed on an inclined plane.

25. Investigate the formula for the Screw with friction, and give the different forms of the equation.

26. State some of the practical advantages and disadvantages of friction.

Solutions of Exercises to Chap. XVII. Page 207.

1. By Art. 310 the coefficient of friction in the first case $= \tan 30^\circ = \frac{1}{\sqrt{3}}$; and in the second case $= \tan 45^\circ = 1$.

2. See Art. 310. $\mu = \cdot 625 = \frac{5}{8} \therefore \tan\phi = \frac{5}{8} \therefore$ the required inclination is the angle whose tangent is $\frac{5}{8}$.

3. Proceeding as in Art. 317, we have

$$P \cos 30^\circ + \mu R - 250 \sin 45^\circ = 0.$$

$$P \sin 30^\circ + R - 250 \cos 45^\circ = 0.$$

$$\mu P \sin 30^\circ + \mu R - \mu 250^\circ \cos 45^\circ = 0.$$

$$\therefore P(\mu \sin 30^\circ - \cos 30^\circ) = 250\ (\mu \cos 45^\circ - \sin 45^\circ).$$

$$\therefore P = 250\,\frac{\mu \cos 45^\circ - \sin 45^\circ}{\mu \sin 30^\circ - \cos 30^\circ} = 250\,\frac{\frac{1}{6} \times \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}}{\frac{1}{6} \times \frac{1}{2} - \frac{1}{2}\sqrt{3}} =$$

$$250\,\frac{\sqrt{2} - 6\sqrt{2}}{1 - 6\sqrt{3}} = 188 \cdot 2 \text{ lbs.}$$

Proceeding in the same way in the second case we should find similar equations but with + sign $\therefore$

$$P = 250\,\frac{\sqrt{2} + 6\sqrt{2}}{1 + 6\sqrt{3}} = 217 \cdot 4 \text{ lbs.}$$

4. See Exam. 3, page 206 Text-Book. Since the angle of inclination of the ladder is 45° the distance from the foot of the ladder to the wall is equal to the vertical distance from the ground to the top of the ladder. Let this distance be d. Let R = the reaction at the foot of the ladder, and R′ that at the top. Let μ $(= \frac{5}{8})$ be the coefficient of friction between the ladder and the ground, and μ' be that between the ladder and the wall. Let W be the weight of the ladder, whose length is $\sqrt{2d^2}$, and suppose the man has just reached the top. Resolve vertically and horizontally and take moments round the foot of the ladder. Then

$$R + \mu' R' - W - \frac{W}{2} = 0 \;\ldots\ldots\ldots\ldots\ldots\;(1)$$

$$\mu R - R' = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(2)$$

$$W \times \frac{d}{2} + \frac{W}{2} \times d - R'd - \mu' R' d = 0 \;\ldots\ldots\;(3)$$

From equation (2) $R' = \mu R$ and $R = \frac{R'}{\mu}$. Substituting in (1) $\frac{R'}{\mu} + \mu' R' = \frac{3W}{2}$ $\therefore R' = \frac{3W\mu}{2(1 + \mu\mu')}$.

From equation (3) $R' = \frac{W}{1 + \mu'}$, $\therefore \frac{W}{1 + \mu'} = \frac{3W\mu}{2(1 + \mu\mu')}$ $\therefore \mu' = \frac{2 - 3\mu}{\mu} = \frac{2 - 3 \times \frac{5}{8}}{\frac{5}{8}} = \frac{1}{5}$, the required coefficient.

5. See Arts. 312 and 310.

$F = \mu R = \mu W \therefore \mu = \frac{F}{W} = \frac{9}{54} = \frac{1}{6}$.

And $\mu = \tan \phi \therefore \phi =$ the angle whose tangent is $\frac{1}{6}$.

6. Let μ be the coefficient of friction, and R the reaction to the plane. Resolving along the plane and at right angles, we have

$\frac{3}{4}W - \mu R - W \sin 30^\circ = 0$.

$R - W \cos 30^\circ = 0$.

Multiplying the last equation by μ and adding, we find

$\frac{3}{4}W - \mu W \cos 30^\circ - W \sin 30^\circ = 0$.

$\therefore \frac{3}{4} - \mu\frac{\sqrt{3}}{2} - \frac{1}{2} = 0 \therefore \mu = \frac{1}{2\sqrt{3}}$.

7. Since the wall is smooth there is no friction at the upper end. Let μ $(= \cdot 5)$ be the coefficient of friction at the lower end. Since the ladder is 10 feet long, and the lower end 6 feet from the the wall, the upper end is 8 feet from the ground. Let $\theta =$ angle of inclination of the ladder; then $\cos \theta = \frac{3}{5}$ and $\sin \theta = \frac{4}{5}$. Let R be the reaction at the lower end, and R′ that at the upper. Let $x =$ the length the man has ascended when the ladder begins to slide.

Resolving vertically and horizontally, and taking moments round the lower end, we find

$W + 3W - R = 0$ . . . . (1)
$\mu R - R' = 0$ . . . . (2)
$3W\, x \cos \theta + W\, 5 \cos \theta - R' 10 \sin \theta = 0$ . . . . (3)

From equations (1) and (2), $R' = \mu R$, $R = 4W \therefore R' = 4\mu W$; substituting in (3) the value of R' and the values of $\cos\theta$ and $\sin\theta$, we have

$3Wx \times \frac{3}{5} + W \times 5 \times \frac{3}{5} - 4 \times \frac{1}{2} \times W \times 10 \times \frac{4}{5} = 0 \therefore x \times \frac{9}{5} = 13 \therefore x = 7\frac{2}{9}$ ft.

8. (*a*) By Art. 310, coefficient of friction $= \tan 30^\circ = \frac{1}{\sqrt{3}}$.

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(*b*) By Art. 312, $F = \mu W = \frac{1}{\sqrt{3}} \times 100 = 57{\cdot}7$ lbs.

9. By Art. 320, the most advantageous angle in this case $= 30^\circ$.

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Let P act at this angle, and let motion be on the point of taking place. Resolving vertically and horizontally, we find

$P \sin 30^\circ + R - 100 = 0$

$P \cos 30^\circ - \mu R = 0$

Multiplying the terms of the first equation by μ and adding to the second, we have

$P(\mu \sin 30^\circ + \cos 30^\circ) = \mu 100 \therefore P = \frac{\mu 100}{\mu \sin 30^\circ + \cos 30^\circ}$

$$= \frac{\frac{1}{\sqrt{3}} \times 100}{\frac{1}{\sqrt{3}} \times \frac{1}{2} + \frac{\sqrt{3}}{2}} = \frac{100}{\frac{1}{2} + \frac{3}{2}} = 50 \text{ lbs., the required force.}$$

10. By Art. 317 when the body is on the point of moving up the plane, $P = \frac{W(\sin i + \mu \cos i)}{\cos\theta + \mu \sin\theta}$. And when the body is on the point of moving down the plane, $P = \frac{W(\sin i - \mu \cos i)}{\cos\theta - \mu \sin\theta}$. These are therefore the extreme limits of the value of P. If P have any intermediate value the forces will be in equilibrium.

11. See Exam. 3, p. 206 Text-Book. Let x = length of ladder ascended when motion is on the point of taking place. Let l = length of ladder; W = weight of ladder; $\frac{W}{2}$ weight of man; R = reaction at lower end; R′ = reaction at upper end; $\mu = \frac{1}{2}$ = coefficient of friction at lower end; $\mu' = \frac{1}{3}$ = coefficient of friction at upper end. Resolve vertically and horizontally, and take moments round the lower end. Then,

$$R + \mu'R' - W - \frac{W}{2} = 0 \quad \ldots\ldots\ldots\ldots (1)$$

$$\mu R - R' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$W\frac{l}{2}\cos 45^\circ + \frac{W}{2}x \cos 45^\circ - \mu'R' \, l \cos 45^\circ$$
$$- R' \, l \cos 45^\circ = 0 \quad \ldots\ldots\ldots\ldots (3)$$

From (2), $\mu R = R' \therefore \frac{1}{2}R = R'$

Substituting in (1), $7R = 9W \therefore R = \frac{9}{7}W$.

Substituting in (3), $W\frac{l}{2} + \frac{W}{2}x - \frac{1}{3} \times \frac{9}{14}Wl - \frac{9}{14}Wl = 0$

$$\therefore \frac{l}{2} + \frac{x}{2} - \frac{3l}{14} - \frac{9l}{14} = 0$$

$\therefore x = \frac{5}{7}l = \frac{5}{7}$ length of ladder.

12. Let F = the force of friction which acts up the plane, and let R = the pressure at right angles to the plane. Resolving along the plane and at right angles, we have

$F - 50 \sin 30^\circ = 0 \therefore F = 50 \times \frac{1}{2} = 25$ lbs.

$R - 50 \cos 30^\circ = 0 \therefore R = 50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3}$ lbs.

13. The resultant reaction is the resultant of the two forces F and R, which are at right angles to each other ∴

Resultant $= \sqrt{25^2 + (25\sqrt{3})^2} = 50$ lbs.

The sine of the angle which this resultant makes with R $= \frac{25}{50} = \frac{1}{2} = \sin 30^\circ \therefore$ the direction of the resultant makes with that of R an angle = to the inclination of the plane, and is therefore vertical.

14. Resolving along the plane,

$F - 100 \sin i = 0 \therefore F = 100 \times \frac{5}{25} = 20$ lbs.

15. Let $P =$ the force. F now acts downwards and is $= 20$ lbs. Resolving along the plane,

$P - 20 - 100 \times \frac{5}{25} = 0 \therefore P = 40$ lbs.

16. By Art. 279, when there is no friction,

$P : W :: \text{height} : \text{length} \therefore \text{height} : \text{length} :: 3 : 5.$

Therefore $\sin i = \frac{3}{5}$, $\cos i = \frac{4}{5}$, and $\mu = \frac{1}{10}$.

With friction, resolving along the plane and at right angles,

$$P - \mu R - 50 \times \tfrac{3}{5} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$R - 50 \times \tfrac{4}{5} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (2) $R = 40$. Substituting in (1) we find $P = 34$ lbs.

Hence any force greater than 34 lbs. will draw the body up the plane.

17. By Art. 315,

$$\frac{P}{W} = \frac{b\sqrt{1+\mu^2} + \mu r}{a\sqrt{1+\mu^2} - \mu r} \therefore \frac{P}{1000} = \frac{24\sqrt{1+\frac{1}{16}} + \frac{1}{4} \times 1}{120\sqrt{1+\frac{1}{16}} - \frac{1}{4} \times 1} \therefore$$

$$\frac{P}{1000} = \frac{24\sqrt{17}+1}{120\sqrt{17}-1} \therefore P = \frac{1000\,(24\sqrt{17}+1)}{120\sqrt{17}-1} = 202{\cdot}5 \text{ lbs.}$$

18. By Art. 255, $P \times 10 = 1000 \times 2 \therefore P = 200$ lbs.

19. Let a vertical section be made passing through the plane and the centre of the parallelopiped. Since the parallelopiped is on the point of sliding, the inclination of the plane = angle of friction = ϕ; and since it is on the point of overturning, the vertical through its C. G. falls on the boundary of its base. It is easy to show by geometry that this vertical makes with the side of the section of the parallelopiped an angle = the inclination of the plane = ϕ. From the centre of the parallelopiped draw a perpendicular

to the base of the parallelopiped, and call this perpendicular x. From the same centre draw a perpendicular to the side of the parallelopiped; this perpendicular $= 1$, since by the question the base $= 2$ feet square. Therefore

$\frac{1}{x} =$ tangent of angle which the vertical makes with the sectional side of parallelopiped $= \tan\phi = \mu$ (Art. 309).

$\therefore \frac{1}{x} = \mu = \frac{1}{2} \therefore x = 2 =$ half the height of the parallelopiped $\therefore$ the height required $= 4$ feet.

20. By Art. 315,

$$P = W\frac{b\sqrt{1+\mu^2}+\mu r}{a\sqrt{1+\mu^2}-\mu r} = 100\frac{2\sqrt{1+\frac{1}{4}}+\frac{1}{2}\times\frac{1}{2}}{10\sqrt{1+\frac{1}{4}}-\frac{1}{2}\times\frac{1}{2}} = 100\frac{\sqrt{5}+\frac{1}{4}}{5\sqrt{5}-\frac{1}{4}}$$

$= 22 \cdot 8$ lbs.

21. By Art. 255, $P \times 10 = 100 \times 2 \therefore P = 20$ lbs.

22. By Art. 315,

$$\frac{P}{W} = \frac{b\sqrt{1+\mu^2}+\mu r}{a\sqrt{1+\mu^2}-\mu r} \therefore W = 20\frac{12\sqrt{1+\frac{1}{4}}-\frac{1}{2}\times\frac{1}{2}}{4\sqrt{1+\frac{1}{4}}+\frac{1}{2}\times\frac{1}{2}} = 20\frac{6\sqrt{5}-\frac{1}{4}}{2\sqrt{5}+\frac{1}{4}}$$

$= 55 \cdot 7$ lbs.

23. By Art. 321, $\frac{P}{W} = \frac{c}{C} \cdot \frac{d+\mu c}{c-\mu d} \therefore \frac{20}{W} = \frac{6}{144} \cdot \frac{1+\frac{1}{4}\times 6}{6-\frac{1}{4}\times 1} \therefore \frac{20}{W} = \frac{6}{144} \times \frac{10}{23} \therefore W = 1104$ lbs.

24. By Art. 287, $P \cdot C = W \cdot d \therefore W = \frac{20 \times 144}{1} = 2880$ lbs.

25. By Art. 313, the acceleration $= g(\sin i - \mu\cos i)$.

(1) $s = \frac{1}{2}ft^2 = \frac{1}{2}g(\sin 60° - \frac{1}{4}\cos 60°) \times 3^2 = \frac{1}{2} \times 32(\frac{\sqrt{3}}{2} - \frac{1}{4}\times\frac{1}{2}) \times 9 = 144\frac{\sqrt{3}}{2} - 144 \times \frac{1}{8} = 72\sqrt{3} - 18$ $= 106 \cdot 7$ feet.

(2) $v = ft = g(\sin i - \mu\cos i)t = 32\left(\frac{\sqrt{3}}{2} - \frac{1}{4}\times\frac{1}{2}\right) \times 3 = 96 \times \frac{\sqrt{3}}{2} - 96 \times \frac{1}{8} = 48\sqrt{3} - 12 = 71 \cdot 1$ feet per 1″.

26. $\phi = 30^\circ \therefore \mu = \tan\phi = \tan 30^\circ = \frac{1}{\sqrt{3}}$.

(1) $$v = ft = g(\sin i - \mu\cos i)t = 32\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} \times \tfrac{1}{2}\right) \times 2$$

$$= 32\left(\frac{3-1}{2\sqrt{2}}\right) \times 2 = \frac{64}{3}\sqrt{3} = 36 \cdot 9 \text{ feet per } 1''.$$

(2) $$s = \frac{u^2}{2f} = \frac{u^2}{2g(\sin i + \mu\cos i)} = \frac{10000}{2 \times 32(\sin 30^\circ + \frac{1}{\sqrt{3}}\cos 30^\circ)}$$

$$= \frac{10000}{64\left(\frac{1}{2} + \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\right)} = \frac{10000}{64} = 156 \cdot 2 \text{ feet.}$$

27. $$s = \frac{u^2}{2g\sin i} = \frac{100^2}{2 \times 32 \times \sin 30} = 312 \cdot 5 \text{ feet.}$$

28. By Arts. 43 and 313, the spaces are

(1) $\frac{u^2}{2g\sin i}$; (2) $\frac{u^2}{2g(\sin i + \mu\cos i)}$.

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